Qualifying Examination of Complex Analysis
Spring 2001
Department of Mathematics
University of California at Riverside

• *The exam is on Saturday, 06/02, 10am – 1pm*
• *Answer any six questions*
• *In each problem, you have to show every step of your calculation.*

1. (a) State the Liouville Theorem.
   (b) Prove that a function which is analytic on the whole plane has a non-essential singularity at \( \infty \) reduces to a polynomial.
2. Let \( \Omega = \mathbb{C} \setminus [-1, 1] \) be the complex plane with the closed interval \([-1, 1]\) on the real axis deleted. Determine and explain which functions below are well-defined analytic functions on \( \Omega \).

(1) \( f(x + iy) = y \);
(2) \( f(z) = \frac{1}{\pi z} \);
(3) \( f(z) = \sqrt{z^2 - 1} \);
(4) \( f(z) = (z^2 - 1)^{\frac{1}{2}} \).
3. Describe how you would construct a conformal map from the domain

$$\Omega = \{ z : \frac{1}{4} \pi < \arg z < \frac{1}{2} \pi \}$$

one to one and onto the exterior of the unit disk. Either give the map explicitly or explain why your idea should work.
4. Prove that if $f(z)$ is an analytic function in the whole complex plane $\mathbb{C}$ with $f^{(n)}(a) = 0$ at a point $a \in \mathbb{C}$ for all $n \geq N$, then $f(z)$ is a polynomial of degree at most $N - 1$. 
5. (a) State the Schwarz lemma of an analytic function \( f : \Delta \rightarrow \Delta \) such that 
\( f(0) = 0 \), where \( \Delta \) is the unit disk.

(b) Let \( g : \Delta \rightarrow \Delta \) be an analytic function (not necessarily \( g(0) = 0 \)), prove 
that \( \frac{|g'(x)|}{1 - |g(x)|^2} \leq \frac{1}{1 - |x|^2} \) for all \( z \in \Delta \).
6. Evaluate the integral
\[ \int_{C} \frac{dz}{z^2(z - 1)(z - 3)}, \]
where \( C \) is the counterclockwise curve \(|z| = 2\).
7 Find the Taylor or Laurent series in power of \( z \) (i.e., with center at the origin) of

\[ f(z) = \frac{1}{(z^2 + 1)(z + 2)} \]

in the regions

1. \( |z| < 1 \);
2. \( 1 < |z| < 2 \);
3. \( |z| > 2 \).
8. (a) State the Montel’s theorem.

(b) Let $\Omega$ be a region and let $M$ be a positive constant. Let $F$ be the family of all analytic functions in $\Omega$ such that $\iint_{\Omega} |f(z)|^2 dx dy < M < +\infty$. Show that $F$ is normal.