We recall some identities and definitions from last time:

If \( v, w \) are vector fields and \( \alpha \) is a \( p \)-form:

1. \( \text{Lie}_v \alpha = i_v d\alpha + d i_v \alpha \)
   
   (\( i_v \) is the interior product and \( d \) is the exterior derivative)

2. \( \text{Lie}_v d\alpha = d \text{Lie}_v \alpha \)

3. \( \text{Lie}_v i_w \alpha = i_{[v,w]} \alpha + i_w \text{Lie}_v \alpha \)

Also from last time:

- We defined a symplectic manifold:

  \( X \) is a smooth \( \mathcal{C}^\infty \)-manifold and \( \omega \) is a 2-form, with \( d\omega = 0 \) (closed) and if \( v \in T_x(X) \) and \( \forall v \in T_x(X) \), \( \omega(v, v) = 0 \) \( \Rightarrow v = 0 \) (non-degeneracy)

- \( \omega \) gives an isomorphism \( T_x \times \mathbb{R} \rightarrow T^*_x \) \( (X) \)

\[
\begin{align*}
\omega : & \ T_x(X) \rightarrow T^*_x(X) \\
& v \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \omega(v, -)
\end{align*}
\]

- Given a function \( f \in \mathcal{C}^\infty(X) \), we defined \( v^*_f \in \text{Vect}(X) \) to be:

\[
\text{df} = -i_{v^*_f} \omega
\]

(Note the sign change)

This \( v^*_f \) is a Hamiltonian vector field.
- Symplectic structure makes $C^\infty(X)$ a Poisson algebra
  (Lie algebra + extra property)

  \[ \{ f, gh \} = \{ f, gh \} + g \{ f, h \} \]

  What is $\{ \cdot, \cdot \}$?

  \[ f, g \in C^\infty(X), \quad \{ f, g \} = \omega(v_f, v_g) \]

  From this definition we get some identities:

  \[ \{ f, g \} = \omega(v_f, v_g) \]

  \[ df = - i_{v_f} \omega = - \omega(v_f, -) \]

  \[ dg = - i_{v_g} \omega = - \omega(v_g, -) \]

  \[ dg(v_f) = - \omega(v_g, v_f) = \omega(v_f, v_g) \]

  \[ \{ f, g \} = dg(v_f) = v_f(g) \]

  \[ \{ f, g \} = i_{v_f} dg \]

  Note: $\text{Lie}_{v_f} g = i_{v_f} dg + d(i_{v_f} g) \quad i_{v_f} g = 0$

  \[ \{ f, g \} = \text{Lie}_{v_f} g \]
Lie algebras

(1) Lie algebra: \( \text{Vect}(X) \)

\[ [\cdot, \cdot] = \text{lie bracket of vector fields} \]

(2) Poisson algebra: \( (C^\infty(X), \{\cdot, \cdot\}) \)

**Proposition:** The set of all Hamiltonian vector fields \( \text{Vect}_H(X) \)

is a Lie subalgebra of \( \text{Vect}(X) \). Furthermore,

\[ C^\infty(X) \xrightarrow{\phi} \text{Vect}_H(X) \]

\[ f \xrightarrow{} v_f \]

is a Lie algebra homomorphism.

**Proof:** Show \( v_f, v_g \in \text{Vect}_H(X) \) \( \Rightarrow \) \( [v_f, v_g] \in \text{Vect}_H(X) \) by showing \( \exists F \in C^\infty(X) \) s.t.

\[ dF = -i_{[v_f, v_g]} \omega \]

From our identities \( \circ \), Liouville's theorem from last time, we have

\[ \text{Lie}_{v_f} i_{v_g} \omega = i_{[v_f, v_g]} \omega + i_{v_g} \text{Lie}_{v_f} \omega = i_{[v_f, v_g]} \omega \]
Note: \( i_{v_g} \omega = -d g \)

\[
\text{Lie}_{v_f} (-d g) = i_{[v_f, v_g]} \omega \\
-d \text{Lie}_{v_f} g = i_{[v_f, v_g]} \omega \\
d \{ f, g \} = -i_{[v_f, v_g]} \omega
\]

So, this implies

\[
\{ f, g \} \rightarrow [v_f, v_g]
\]

which implies

\[
\phi (\{ f, g \}) = [\phi (f), \phi (g)].
\]

\[\square\]

**Question:** What is the kernel of \( \phi \)?

**Answer:** \( f \in \ker \phi \implies v_f = 0 \implies -i_{v_f} \omega = 0 = \omega (0, -) \)

\( \implies df = 0 \)

\( \implies f \) is locally constant.

So, \( \ker \phi = \mathbb{R}^{n_c}, \; n_c = \# \) of connected components of \( X \).
Example: $\mathbb{R}^2$, $\omega = dp \wedge dq$

if $f \in \mathcal{C}^\infty(\mathbb{R}^2)$, what is $v_f$?

$$v_f = v_f^q \frac{d}{dq} + v_f^p \frac{d}{dp}$$

$$df = \frac{df}{dq} dq + \frac{df}{dp} dp$$

$$df = -\omega(v_f, -)$$

$$= (-v_f^q \frac{d}{dq} dp \wedge dq) - (v_f^p \frac{d}{dp} dp \wedge dq)$$

$$= v_f^q \frac{d}{dq} dp \wedge dq - v_f^p dq$$

$$= v_f^p dp - v_f^p dq$$

Equating coefficients, we have

$$\frac{df}{dq} = -v_f^q$$

$$\frac{df}{dp} = v_f^p$$

In $\mathbb{R}^2$,

$$v_f = \frac{df}{dp} \frac{d}{dq} - \frac{df}{dq} \frac{d}{dp}$$

Then,

$$\{f, g\} = v_f(g) = \frac{df}{dp} \frac{dg}{dq} - \frac{df}{dq} \frac{dg}{dp}$$
Hamiltonian mechanics:

\[ F = ma, \quad a = \frac{d^2 q}{dt^2} \]

\[ F = m \frac{d^2 q}{dt^2}, \quad 2^{nd} \text{ order ODE} \]

Let the force be minus the gradient of a potential function

\[ V: \mathbb{R}^n \longrightarrow \mathbb{R} \]

\[ -\nabla V = m \frac{d^2 q}{dt^2} \]

• define momentum \( p = mv = m \frac{dq}{dt} \)

• 2 first order equations:

\[ \frac{dq}{dt} = \frac{p}{m}, \quad \frac{dp}{dt} = -\nabla V \]

Define \( H: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R} \)

\[ H(q, p) = \frac{p^2}{2m} + V(q) \]

↑

↑

Kinetic Potential
\[ \frac{\partial H}{\partial q} = \nabla V, \quad \frac{\partial H}{\partial p} = \frac{p}{m} \]

\[ \Rightarrow \quad \frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q} \]

These are Hamilton's equations.

(a solution is: \(q(t), p(t)\))

Constructing a symplectic structure:

Let \(\mathbb{R}^2\) be our configuration space. This implies that \(\mathbb{R}^2\) is the space of position and momentum, \((q, p) \in \mathbb{R}^2\) (phase space).

The symplectic form is \(\omega = dp \wedge dq\), i.e., the Hamiltonian \(H \in C^0(\mathbb{R}^2)\).

What is \(\nabla_H\) ?

\[ \nabla_H = \frac{\partial H}{\partial p} \frac{\partial}{\partial q} - \frac{\partial H}{\partial q} \frac{\partial}{\partial p} \]
What is the flow of $\nu_H$?

Find a curve $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$

$t \rightarrow (q(t), p(t))$

Then,

\[
\frac{d\gamma}{dt} = \frac{dq}{dt} \frac{\partial}{\partial q} + \frac{dp}{dt} \frac{\partial}{\partial p}
\]

If $\nu_H$ is the vector field tangent to $\gamma$, then

\[
\frac{dq}{dt} = \frac{\partial H}{\partial t}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}
\]

Physical meaning of $\lbrace \cdot, \cdot \rbrace$:

$f \in C^0(\mathbb{R}^2)$, what is $\lbrace H, f \rbrace$?

$\lbrace H, f \rbrace = \text{Lie}_{\nu_H} f = \frac{df}{dt}(q(t), p(t))$

Conserved quantities:

"conservation of energy"

\[
\frac{dH(q(t), p(t))}{dt} = 0, \quad \text{which we have since } \lbrace H, H \rbrace = 0.
\]