Rational curves in $\mathbb{P}^n$

The purpose of this note is to review some notations and results, both qualitative and enumerative, about rational curves in $\mathbb{P}^n$. Proofs and further details may be found in [R2, R3, R4] and references therein.

We begin by reviewing some qualitative results about families of rational curves in $\mathbb{P}^n$, especially for $n = 2$ or 3. See [R2][R3] [R4] and references therein for details and proofs. In what follows we denote by $\bar{V}_d$ the closure in the Chow variety of the locus of irreducible nonsingular rational curves of degree d in $\mathbb{P}^n, n = 3$, with the scheme structure as closure, i.e. the reduced structure (recall that the Chow form of a reduced 1-cycle Z is just the hypersurface in $G(1, \mathbb{P}^3)$ consisting of all linear spaces meeting Z); if $n = 2$ $\bar{V}_d$ is just the (closure of) the Severi variety. Thus $\bar{V}_d$ is irreducible reduced of dimension

$$\dim(\bar{V}_d) = 4d, n = 3$$
$$\dim(\bar{V}_d) = 3d - 1, n = 2.$$

Let

$$A_1, \dots , A_k \subset \mathbb{P}^n$$

be a generic collection of linear subspaces of respective codimensions $a_1, \dots , a_k$, $1 \leq a_i \leq n$ (so if $n = 2$ these are just lines and points) We denote by

$$B = B_d = B_d(a.) = B_d(A.)$$

the normalization of the locus (with reduced structure)

$$\{(C, P_1, \dots , P_k) \ : \ C \in \bar{V}_d, P_i \in C \cap A_i, i = 1, \dots , k\}.$$

If all $a_i > 1$ then this locus is also the normalization of its projection to $\bar{V}_d$, i.e. the locus of degree-d rational curves (and their specializations) meeting $A_1, \dots , A_k$. We have

(A1) $$\dim B = (n + 1)d + n - 3 - \sum(a_i - 1).$$

When $\dim B = 0$ we set

(A2) $$N_d(a.) = \deg(B).$$

Note that $N_d(1, a.) = dN_d(a.)$, which allows us to reduce the computation of the general $N_d(a.)$ to the case where all $a_i > 1$, in which case we will say the condition-vector $(a.)$ is *proper* When $n = 2$ and $(a.)$ is proper, then all the $a_i = 2$ so they will be dropped from the notation. The number k of a_i such that $a_i > 1$ is called the *length* of the condition-vector $(a.)$. Whenever $b = \dim(B) \geq 0$, it is convenient to set

(A3) $$N'_d(a.) = N_d(a., b + 1)$$

and note that this is the degree in $\mathbb{P}^n$ of the locus swept out by the curves in $B(a.)$.

The numbers N_d and $N_d(a.)$, first computed in general by Kontsevich and Manin , are computed in [R2],[R3] by an elementary method, reviewed below, based on recursion on d and k.

Now suppose $B = B(a.)$ is such that $\dim B = 1$ and $(a.)$ is proper and let

$$\pi : X \to B$$

be the normalization of the tautological family of rational curves, and

$$f : X \to \mathbb{P}^n$$

the natural map. The following summarizes results from [R2][R3][R4] :

Theorem A0. *(i) X is smooth .*

(ii) Each fibre C of π is either

(a) a $\mathbb{P}^1$ on which f is either an immersion with at most one exception which maps to a cusp ($n = 2$) or an embedding ($n > 2$); or

(b) a pair of $\mathbb{P}^1$'s meeting transversely once, on which f is an immersion with nodal image ($n = 2$) or an embedding ($n > 2$); or

(c) if $n = 3$, a $\mathbb{P}^1$ on which f is a degree-1 immersion such that $f(\mathbb{P}^1)$ has a unique singular point which is an ordinary node.

(iii) If $n > 2$ then $\bar{V}_{d,n}$ is smooth along the image $\bar{B}$ of B, and $\bar{B}$ is smooth except, in case some $a_i = 2$, for ordinary nodes corresponding to curves meeting some A_i of codimension 2 twice. If $n = 2$ then $\bar{V}_{d,n}$ is smooth in codimension 1 except for a cusp along the cuspidal locus and normal crossings along the reducible locus, and $\bar{B}$ has the singularities induced from $\bar{V}_{d,n}$ plus ordinary nodes corresponding to curves with a node at some A_i, and no other singularities.

Next, we review some of the enumerative apparatus introduced in [R3][R4] to study X/B. Set

$$m_i = m_i(a.) = -s_i^2, i = 1, ..., k. \tag{A4}$$

Note that if $a_i = a_j$ then $m_i = m_j$; in particular if $n = 2$ (and $(a.)$ is proper, as we are assuming), they are all equal and will be denoted by m_d, i.e. for $n = 2$,

$$m_d = m_1(2^{3d-2}).$$

It is shown in [R2] [R3][R4] that these numbers can all be computed recursively in terms of data of lower degree d and lower length k. For instance for $n = 2$ we have

$$2m_d = \sum_{d_1+d_2=d} N_{d_1} N_{d_2} d_1 d_2 \binom{3d-4}{3d_1-2}. \tag{A5}$$

For $n \geq 2$, note that

$$s_i.s_j = N_d(..., a_i + a_j, ..., \hat{a}_j, ...), i \neq j.$$

(so for $n = 2$ this is always 0 if $(a.)$ is proper). Also, letting R_ℓ denote the sum of all fibre components not meeting s_ℓ , we have

$$s_j \cdot R_\ell = \sum N'_{d_1}(a_i : i \in I) N'_{d_2}(a_i : i \notin I). \tag{A6}$$

the summations being over all $d_1 + d_2 = d$ and all index-sets I with $j \in I, \ell \notin I$.

So all these numbers may be considered known. Then we have

$$m_i = \frac{1}{2}(s_i.R_j + s_i.R_p - s_j.R_p) - s_i.s_j - s_i.s_p + s_j.s_p$$

for any distinct i, j, p, and the RHS here is an expression of lower degree and/or length, hence may be considered known.

Next, set

$$L = f^*(\mathcal{O}(1)),$$

and note that

$$L^2 = N_d(2, a.),\ L.s_i = N_d(a_1, ..., a_i + 1, ...),\ i = 1, ..., k$$

(in particular, $L.s_i = 0$ if $a_i = n$.) We computed in [R3] that, for any i,

$$L \sim ds_i - \sum_{F\in\mathcal{F}_i} \deg(F)F + (N_d(a_1, \dots, a_i + 1, \dots) + dm_i(a.))F_0$$

where F_0 is the class of a complete fibre and $\mathcal{F}_i$ is the set of fibre components not meeting s_i. Consequently we have

$$N_d(2, a_1, \dots) = 2dN_d(a_1 + 1, a_2, \dots) + d^2 m_1(a.) - \sum_{F\in\mathcal{F}_1(a.)} (\deg F)^2 \tag{A7}$$

and clearly the RHS is a lower degree/length expression, so all the $N_d(2, \dots)$ are known. We also have for $n > 2$ that

$$N_d(a_1, a_2 + 1, ...) - N_d(a_1 + 1, a_2, ...) =$$

$$dN_d(a_1 + a_2, ...) - \sum_{F\in(\mathcal{F}_1-\mathcal{F}_2)(a.)} (\deg F) + N_d(a_1 + 1, a_2, ...) + dm_1(a.) \tag{A8}$$

and again the RHS here is 'known', hence so is the LHS, which allows us to 'shift weight' between the a_i's till one of them becomes 2, so we may apply (A7), and thus compute all of the $N_d(a.)$'s.

Next, it is easy to see as in [R3] that

$$L.R_i = \sum_{d_1+d_2=d} \binom{3d-2}{3d_1-1} d_1 d_2^2 N_{d_1} N_{d_2},\ n = 2 \tag{A9}$$

(in this case this is independent of i and we will just write it as $L.R$);

$$L.R_j = \sum d_2 N'_{d_1}(a_i : i \in I) N_{d_2}(a_i : i \notin I),\ n \geq 2 \tag{A10}$$

the summation for $n > 2$ being over all $d_1 + d_2 = d$, and all index-sets I such that $j \in I$.

Finally, the relative canonical class $K_{X/B} = K_X - \pi^*(K_B)$ was computed in [R3] as

$$K_{X/B} = -2s_i - m_i F + R_i \tag{A11}$$

for any i. Note that $-R_i^2$ equals the number of reducible (equivalently, singular) fibres in the family X/B, a number we denote by $N_d^{\text{red}}(a_.)$, and which is easily computable by recursion, namely

$$N_d^{\text{red}} = \sum_{d_1+d_2=d} \binom{3d-2}{3d_1-1} d_1 d_2 N_{d_1} N_{d_2}, n = 2, \tag{A12}$$

$$N_d^{\text{red}}(a_.) = \sum N'_{d_1}(a_i : i \in I) N'_{d_2}(a_i : i \notin I), n \geq 2, \tag{A13}$$

where the latter sum extends over all $d_1 + d_2 = d$ and all index-sets I.

From this we compute easily that

$$L.K_{X/B} = -2N_d(...a_i + 1...) - dm_i + L.R_i, \tag{A14}$$

$$K_{X/B}^2 = -N_d^{\text{red}}(a_.). \tag{A15}$$

References

[R2] Z. Ran, *Bend, break and count*, Isr. J. Math **111** (1999), 109-124.
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