APPENDIX A: THE (-1) TWIST

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ABSTRACT. We prove a nonvanishing result for the (-1) twist of the canonical bundle of a subvariety of a generic hypersurface in projective space.

The purpose of this appendix is to extend the main result of the body of the paper, henceforth referred to as [CR], to cover the case of the (-1) twist of the canonical bundle. A similar result in a more restricted range was obtained independently by Pacienza (math.AG/0204297). Our result is as follows.

Theorem A1. *Let*

$$X \in \mathcal{L}_d$$

be generic with

$$d(d-2)/8 \geq 3n-1-k, d > n$$

and

$$f : Y \to X$$

a desingularization of an irreducible subvariety of dimension k. Set

$$t = \max(-1, -d+n+1+[\frac{n-k}{2}]).$$

Then either

$$h^0(\omega_Y(t)) > 0$$

or $f(Y)$ is contained in the union of the lines lying on X.

In the case where $k = n-3$ we have furthermore that if $h^0(\omega_Y(t)) = 0$, then $f(Y)$ is ruled by lines.

To give an idea as to the significance of the conclusion, we just note that whenever $h^0(\omega_Y(-1)) \neq 0$, Y is of general type; in fact, the canonical system on Y yields a rational map that is birational to its image. Moreover,

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$p_g(Y) - 1$ is bounded below by the dimension of the projective span of Y in $\mathbb{P}^n$.

The proof of Theorem A1 is largely identical to that in [CR], so we will just indicate the differences. Assuming

$$h^0(\omega_Y(-1)) = 0,$$

we wish to show that Y is contained in the union of lines in X. Compared to [CR], display (6.3), we may now assume only

$$-d + n + 1 + s \geq 0,$$

then Remark 6.3.1 allows us to assume that in general $\ell(y)$ is a 'bicontact line' to $X_{F(y)}$, i.e. meets it in 2 points with multiplicities $r, d-r$ for some $1 \leq r \leq d-1$. This gives rise to a (GL(n)-equivariant, hence submersive) map

$$\mathcal{Y} \to \Delta_{r,d-r}$$

where $\Delta_{r,s}$ is the closure of

$$\{(\ell, x, x', F) : x \neq x', \ell.X_F \geq rx + sx'\} \subset G \times \mathbb{P}^n \times \mathbb{P}^n \times S.$$

Note that $\Delta_{r,s}$ is a vector bundle over the double incidence variety

$$I^2 := \{(\ell, x, x') : x, x' \in \ell\},$$

and in particular is smooth. Below we shall compute the canonical bundle of $\Delta_{r,s}$. Once this is done, the proof may be concluded by copying the arguments of [CR], beginning with Lemma 6.4.

We shall assume henceforth that

$$r \geq s.$$

Consider the product

$$\Delta_r \times \mathbb{P}^n,$$

with line bundle

$$\mathcal{O}_2(-1) := p_2^*(\mathcal{O}(-1)).$$

Then the zero-scheme $\Delta_{r,0}$ of the natural map

$$\mathcal{O}_2(-1) \to Q$$

is just the locus

$$\{(\ell, x, x', F) : (\ell, x, F) \in \Delta_r, x' \in \ell\}$$

and admits a natural map to I^2.

Clearly, we have

$$\omega_{\Delta_{r,0}} = \omega_{\Delta_r} \otimes \mathcal{O}_G(1) \otimes \mathcal{O}_2(-n+1).$$

Now consider on on $\Delta_{r,0}$ the zero-scheme of the natural map

$$\mathcal{O}_S(-1) \to \mathcal{O}_2(d).$$

This consists of $\Delta_{r,1}$ plus the pullback of the diagonal divisor

$$D = \{(\ell, x, x)\} \subset I^2.$$

Since D is the degeneracy locus of the natural map

$$\mathcal{O}_1(-1) \oplus \mathcal{O}_2(-1) \to \mathcal{S},$$

it is easy to see that

$$\mathcal{O}(D) = \mathcal{O}_G(-1) \otimes \mathcal{O}_1(1) \otimes \mathcal{O}_2(1),$$

hence by the adjunction formula we have

$$\omega_{\Delta_{r,1}} = \omega_{\Delta_{r,0}} \otimes \mathcal{O}_G(1) \otimes \mathcal{O}_1(-1) \otimes \mathcal{O}_2(d-1).$$

Now we can argue as above, considering the natural injection

$$\mathcal{O}_2(-1) \to \mathcal{S}$$

and the induced filtration $F_2^{\cdot}$ on $\mathrm{Sym}^d(\mathcal{S}^{\mathrm{v}})$. The zero scheme of the natural map

$$\mathcal{O}_S(-1) \to F_2^1/F_2^s$$

consists of $\Delta_{r,s}$ plus D, i.e. $\Delta_{r,s}$ is a zero scheme of $(F_2^1/F_2^s)(-D)$, and therefore

$$\omega_{\Delta_{r,s}} = \omega_{\Delta_{r,1}} \otimes \det(F_2^1/F_2^s) \otimes \mathcal{O}(-(s-1)D).$$

This finally gives the formula (valid for $r \geq s$)

$$\omega_{\Delta_{r,s}} =$$

$$\mathcal{O}_G\left(\frac{r(r-1)+s(s-1)}{2}+s+1-n\right)\otimes\mathcal{O}_1(r(d-r)-n+r-s+1)\otimes\mathcal{O}_2(s(d-s)-n+1).$$

Remark. At the risk of stating the obvious, let us point out that the present method seems to shed no light on any twists of ω_Y strictly below $\omega_Y(-1)$.

APPENDIX B: A GENERAL POSITION LEMMA

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The purpose of this appendix is to give an alternate and somewhat shorter proof of the main Lemma (Lemma 1.2) of [CLR]. This proof, while not germane to the main body of the present paper, is close to it in spirit and the author hopes that its publication here would be a further step forward in the process of unifying the various methods and results available in the subject, a process hopefully also contributed to by the main paper itself. The statement goes as follows.

Lemma B1. *Let $Y \subset \mathbb{P}^{n+1}$ be an irreducible subvariety spanning a $\mathbb{P}^{p+1}$. Fix integers r, k with $0 \leq r-1 \leq k \leq d$. Let $\mathcal{L}_d \to \mathbb{C}^N$ be a linear map of vector spaces such that for $y_1, \dots, y_{r-1}$ general points of Y, the restriction to $\mathcal{L}_d(-y_1 - \dots - y_{r-1})$ induces a surjection*

$$\Psi : \mathcal{L}_d(-y_1 - \dots - y_{r-1}) \to \mathbb{C}^{k+1}.$$

Then for a general choice of elements $h_{k+1}, \dots, h_d \in \mathcal{L}$ and for general subsets $Y_1, \dots, Y_k \subset Y$ each of cardinality p, with $Y_i \ni y_i$, $i = 1, \dots, r-1$, the restriction of Ψ to the subspace $\mathcal{L}(-Y_1) \cdots \mathcal{L}(-Y_k) h_{k+1} \cdots h_d$ surjects.

proof. By a *good chain* in $\mathbb{P}(\mathcal{L}_1) = (\mathbb{P}^{n+1})^v$ we mean a (connected) chain whose components are straight lines (i.e. pencils) of the form $\mathcal{L}(-S)$ where S is a general p-tuple in Y and whose 'vertices' (i.e. singular points) are general in $\mathbb{P}(\mathcal{L}_1)$. Clearly two general elements of $\mathcal{L} = \mathcal{L}_1$ can be joined by a good chain. Let $\mathcal{M}_d \subset \mathcal{L}_d$ be the set of monomials. By a *good chain* in $\mathcal{M}_d$ we mean a chain which is a union of subchains of the form $\mathcal{C}'_i = h_1 \cdots \mathcal{C}_i \cdots h_d$ where $\mathcal{C}_i$ is a good chain in $\mathcal{L}$. It is easy to see that two general monomials can be joined by a good chain.

Next, let us say that a monomial $h_{d-e+1} \cdots h_d \in \mathcal{M}_e$ is rel $\bar{y}, \bar{y} = \{y_1, ..., y_{r-1}\}$ if $h_i \in \mathcal{L}(-y_i), i = d-e+1, ..., r-1$ (this condition is vacuous if $r-1 < d-e+1$); denote by $\mathcal{M}_e(-\bar{y})$ the set of these. Again it is easy to see that two general elements of $\mathcal{M}_e(-\bar{y})$ can be joined by a good chain within $\mathcal{M}_e(-\bar{y})$.

Now to prove the Lemma it suffices to show by induction on $q, 0 \leq q \leq k$, that, with the above notations,

$$\dim \Psi(\mathcal{L}(-Y_1) \cdots \mathcal{L}(-Y_q) h_{q+1} \cdots h_d) \geq \min(q+1, k+1)$$

(for general choices rel $\bar{y}$). For $q = 0$ this is clear as Ψ is nonzero on a general element of $\mathcal{M}_d(-\bar{y})$, because these span $\mathcal{L}_d(-\bar{y})$. Assume it is true for q and false for $q+1$, and suppose first that $q+1 \leq r-1$. Now because

$$Z := \Psi(\mathcal{L}(-Y_1) \cdots \mathcal{L}(-Y_q) h_{q+1} \cdots h_d)$$

$$= \Psi(\mathcal{L}(-Y_1)\cdots\mathcal{L}(-Y_q)\mathcal{L}(-Y_{q+1})h_{q+2}\cdots h_d)$$

by assumption (i.e Z doesn't move as h_{q+1} varies in $\mathcal{L}(-Y_{q+1})$, and because two general elements of $\mathcal{L}(-y_{q+1})$ can be joined by a good chain (whose components are of the form $\mathcal{L}(-Y_{q+1})$), it follows that Z is independent of $h_{q+1} \in \mathcal{L}(-y_{q+1})$, fixing the other h's. Since $y_1, ...y_{r-1}$ are all interchangeable, it follows that a similar statement holds for any permutation of them. In particular Z contains all pencils of the form $\Psi(h_1\cdots\mathcal{L}(-Y_j)\cdots h_d), j = 1, ..., r-1$ and from connectivity of $\mathcal{M}_{r-1}(-\bar{y})$ by good chains it follows that in fact $Z = \Psi(\mathcal{L}_{r-1}(-\bar{y})h_r\cdots h_d)$. As for the remaining h_j's, say $j = r$, we may pick $y_r \in f^*(h_r)_0$ and apply similar reasoning to $y_1, ..., y_{r-1}, y_r$ in place of $y_1, ..., y_{r-1}$. The foregoing argument yields that Z is independent of $h_r \in \mathcal{L}(-y_r)$, fixing y_r and the other h's (indeed that $Z = \Psi(L_r(-\bar{y} - y_r)h_{r+1}\cdots h_d)$. Then another similar argument with good chains of h_j's not fixing y_j yields easily that Z is actually independent of $h_j \in \mathcal{L}$. Now since we can connect two general elements of $\mathcal{M}_d(-\bar{y})$ by a good chain, we conclude that $Z = \Psi(\mathcal{L}_d(-\bar{y}))$, which is a contradiction.

The case $q + 1 \geq r$ is similar but simpler: we may conclude directly that Z is independent of $h_{q+1} \in \mathcal{L}(-Y_{q+1})$, hence of $h_j \in \mathcal{L}(-Y_j)$ for all $r \leq j \leq k$ and use good chains as above to deduce a contradiction. $\square$

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