ABSTRACT. We consider a general fibre of given length in a generic projection of a variety. Under the assumption that the fibre is of local embedding dimension 2 or less, an assumption which can be checked in many cases, we prove that the fibre is reduced and its image on the projected variety is an ordinary multiple point.

The study of generic linear projections of a smooth projective variety $X \subset \mathbb{P}^N$, especially their singularities, was a favorite topic of classical projective geometers for generations (see e.g. [9]), and these projections remain among the main sources of finite, non-flat morphisms in Algebraic Geometry. Modern interest in this area seems to begin with Mather [5], whose work was reinterpreted in algebro-geometric language by Alzati- Ottaviani [1]. One may formulate a folklore ‘generic projection conjecture’ to the effect that the projection of $X$ from a generic linear subspace $\Lambda \in \mathbb{P}^N$ has only the ‘expected’ singularities. After numerous partial results including [7], [2], the conjecture was recently proven for projections from a point by Gruson and Peskine [4] (which the reader may consult for further introductory comments, references, as well as examples and applications).

In [8], we gave various extensions of the Gruson-Peskine Theorem, including one about projections from lines. More generally, we showed inter alia that the locus of fibres of given length $k$ of a generic projection from a center $\Lambda$ of any dimension $\lambda$ is smooth of the expected dimension, just as long as the fibres have local embedding dimension 2 or less. The latter hypothesis is of course automatic for $\lambda = 0, 1$, but as we shall see below (cf. Lemma 1 and Example 3), it also holds in a fair range of cases beyond that.

One question which was left open in [8] was as to the nature of the general fibre of given length $k$ and embedding dimension $\leq 2$: was it in fact in fact reduced, i.e. a collection of distinct points; even more optimistically, are those in general position relative to the $(\lambda + 1)$-plane they are constrained to lie on? The purpose of this paper is to settle these questions. We will settle the first question affirmatively, and show further that the resulting $k$-fold point on the projected image is ‘ordinary’ or ‘transverse’ (see below for definitions). The second question will be settled negatively by a counterexample (see Example 6). Such counterexamples seem rare though, and the possibility remains
that they could be effectively characterized in terms of low-degree defining equations, though we don’t have a specific conjecture—much less, result, to offer.

We begin with some notation. Let

\[ X \subset \mathbb{P}^m \]

be an irreducible closed subvariety of codimension \( c \). Let

\[ \Lambda = \mathbb{P}^\lambda \subset \mathbb{P}^m, \lambda < c \]

be a generic linear space. Let

\[ X^\Lambda \subset \mathbb{P}^{m-\lambda-1} \]

be the (locally closed) locus of \((\lambda + 1)\)-planes \( L \) disjoint from the singular locus of \( X \), containing \( \Lambda \) and meeting \( X \) in a scheme of length \( k \) or more. Now consider the following condition on \( L \in X^\Lambda \):

(EDIM2) The local embedding dimension of \( Z = L \cap X \) is 2 or less everywhere.

Under this condition, it was proven in [8] that \( X^\Lambda \) is smooth of the expected codimension in \( \mathbb{P}^{m-\lambda-1} \), viz. \( k(c - \lambda - 1) \), at \( L \). The tangent space to \( X^\Lambda \) at a point \( L \) can be identified with \( H^0(N^s) \) where \( N^s \) is a certain subsheaf of the normal bundle \( N_{L/\mathbb{P}^m} \) which differs from it only at points of \( Z = L \cap X \), and at those points consists grosso modo of the vectors normal to \( L \) and tangent to \( X \).

Before stating the new result, we make a remark about the EDIM2 condition.

**Lemma 1.** Notations as above, condition (EDIM2) holds for all \( L \in X^\Lambda \) provided either

(i) \( \min(\lambda + 1, m - c) \leq 2 \); or

(ii) \( m < 4c - 3\lambda + 6 \).

**Proof.** (i) is trivial. As for (ii), assuming \( m - c \geq 3, \lambda > 1 \), note that if \( Z \) has embedding dimension 3 or more at \( p \), then the embedded tangent space \( \mathring{T}_pX \) meets \( L \) in at least a 3 (projective)-dimensional space, hence meets \( \Lambda \), a hyperplane in \( L \), in at least a 2-dimensional space. The space of pairs \((p, U)\), where \( U \) is a 2-dimensional subspace of \( \mathring{T}_pX \), is \( (4(m - c) - 6) \)-dimensional, and it is \( 3(m - \lambda) \) conditions for \( U \) to be contained in \( \Lambda \). Since \( \Lambda \) is general with respect to \( X \), the Bertini theorem on transversality of a general translate yields the result. \( \square \)

As a matter of terminology, a collection of subspaces \( A_i \) of a vector space \( B \) is said to be *transverse* if

\[ \text{codim}(\bigcap A_i, B) = \sum \text{codim}(A_i, B) \]

A point \( y \) on a subvariety \( Y \) in a smooth variety \( P \) is said to be a *transverse* \( k \)-fold point if locally at \( y \), \( Y \) is a union of \( k \) smooth branches \( Y_1, ..., Y_k \) and \( T_yY_i, i = 1, ..., k \) is a transverse collection of subspaces of \( T_yP \).
Theorem 2. Notations as above, assume
(a) $L$ is general in $X^\Lambda$;
(b) $c > \lambda + 1$;
(c) condition (EDIM2) holds.
Then
(i) $Z$ is reduced;
(ii) each point of $Z$ is general on $X$;
(iii) $Z$ projects to a transverse $k$-fold point on the projection $\pi_\Lambda(X)$.

Note that in (ii), it is not claimed that $Z$ is a generic $k$-tuple on $X$, only each point in itself is generic. Thus given any subvariety $Y \subsetneq X$, $Z$ may be assumed disjoint from $Y$ by taking $\Lambda$ general enough.

Example 3. For $m - c = 3$, i.e. $X$ a 3-fold, and $\lambda = 2$, Lemma 1 shows that condition (EDIM2) is weaker than $c > \lambda + 1 = 3$. Therefore the theorem is applicable to the generic projection of any smooth 3-fold $X \subset \mathbb{P}^m, m \geq 7$, to $\mathbb{P}^{m-3}$. In particular, for $m = 7$, the projected image of $X$ in $\mathbb{P}^4$ has finitely many transverse 4-fold points, each with 4 linearly independent tangent hyperplanes.

In fact, for $\lambda = 2$, i.e. projection from a plane, and $X$ of codimension $c = 4$, the theorem applies for $X$ of dimension up to 11, i.e. for projection from $\mathbb{P}^m$ to $\mathbb{P}^{m-3}$ for $m \leq 15$.

The proof of the Theorem is a continuation of the argument used to prove Theorem 5.1 in [8]. A key role is played again by the vanishing lemma ([8], Lemma 5.8) for sub-sheaves of the normal bundle $N_{\Lambda}$ whose sections move with $\Lambda$ (which, we recall, is a generic $\lambda$-plane). Originally, the Lemma was applied to the secant sheaf $N^s$. Here, we note that it can be applied as well to suitable subsheaves of $N^s$, and exploit the consequences. Incidentally, we neglected to mention in [8] that the vanishing lemma implies a regularity result of the secant sheaves, and this omission will be rectified below.

Because of the key role it plays in this paper, we proceed to state without proof a special case, sufficient for our purposes, of the Lemma in question. This concerns the following data:
- $\Lambda$, a $\mathbb{P}^\lambda$ in $\mathbb{P}^m$ that is generic with respect to the subvariety $X$ as above (in particular, $X \cap \Lambda = \emptyset$),
- $L$, an arbitrary $\mathbb{P}^{\lambda+1}$ containing $\Lambda$,
- $M \subset N_{L/\mathbb{P}^m}$, a coherent subsheaf of the normal bundle, such that the support of $N_{L/\mathbb{P}^m} / M$ finite and disjoint from $\Lambda$, and having the property that $H^0(M)$ ‘moves infinitesimally with $\Lambda$’, in the sense that given any $v_1 \in H^0(N_{\Lambda/\mathbb{P}^m})$, there exists $v_2 \in H^0(M)$ having the same image as $v_1$ in $H^0(M \otimes \mathcal{O}_\Lambda)$.

Vanishing Lemma (see Lemma 5.8 of [8]). Assumptions as above, we have $H^1(M(t)) = 0$ for $t \geq -1$.

The proof is given in [8] and will not be reproduced here.
Proof of Theorem. We fix a sufficiently general $L$ as in the Theorem. Then by [8], Theorem 5.1, we may assume that $Z = L \cap X$ has length exactly $k$. Pick any $p \in Z$.

Now recall the ‘secant subsheaf’ $N^s \subset \mathcal{H}om(I_L, O_L)$ defined in [8], which fits in an exact diagram

\[
\begin{array}{cccc}
0 & \rightarrow & \mathcal{H}om(I_Z/I_X, O_Z) & \rightarrow 0 \\
\downarrow & \downarrow & \downarrow & \\
N^s & \rightarrow & \mathcal{H}om(I_L, O_L) & \rightarrow 0 \\
\downarrow & \downarrow & \downarrow & \\
\mathcal{H}om(I_L, O_L) & \rightarrow & \mathcal{H}om(I_L, O_Z) & \rightarrow 0 \\
\downarrow & \downarrow & \downarrow & \\
\mathcal{H}om(I_L \cap I_X, O_Z) = \mathcal{H}om(I_L \cap I_X, O_Z)
\end{array}
\]

(note $I_Z/I_X = (I_L + I_X)/I_X = I_L/(I_L \cap I_X)$). Of course, $\mathcal{H}om(I_Z/I_X, O_Z)$ is just a skyscraper sheaf with stalk equal to the tangent space at $Z$ to the punctual Hilbert scheme of $X$, that is $X^{[k]}$ where $k = \ell(Z)$. Now we have a support decomposition

\[
\mathcal{H}om(I_Z/I_X, O_Z) = \bigoplus T_p, \ T_p = \mathcal{H}om(I_Z/I_X, O_Z)_p
\]

which corresponds to a local analytic decomposition

\[
X^{[k]} = \prod X^{[k(p)]}, k(p) := \ell(p(Z)).
\]

via the identification of Zariski tangent spaces

(2) $T_{Z_p}X^{[k(p)]} = H^0(T_p)$.

It was observed in [8]- and of course is well known- that the assumption of embedding dimension 2 or less implies, via Fogarty’s theorem on the smoothness of the Hilbert scheme of a smooth surface [3], that $X^{[k(p)]}$ is smooth at $Z_p$ and its general point corresponds to a reduced scheme.

Now I claim that the surjection

\[
N^s \rightarrow T_p
\]

(cf. (1)) induces a surjection on $H^0$:

(3) $H^0(N^s) \rightarrow H^0(T_p)$.

If this holds, then the natural (local) map from the secant scheme $X^k$ to $X^{[r(p)]}$ is smooth, hence surjective, near $Z$. Therefore $Z_p$ is general on $X^{[r(p)]}$, hence reduced. Since $Z_p$ is supported at $p$, it follows that $Z_p = p$ is a single reduced general point of $X$, this proving parts (i) and (ii) of the Theorem.

To prove surjectivity of (3), note that its cokernel is anyhow of the form $H^0(A)$ for some quotient $A$ of the skyscraper sheaf $T_p$, such that the natural map $H^0(N^s) \rightarrow H^0(A)$
vanishes. Let \( N^s_0 \subset N^s \) be the corresponding subsheaf of \( N^s \) under the surjection \( N^s \to T_p \). Thus we have an exact sequence
\[
0 \to N^s_0 \to N^s \to A \to 0. \tag{4}
\]
Then by the generality of \( L \), \( N^s_0 \) satisfies the hypotheses of our Vanishing Lemma, and hence \( H^1(N^s_0) = 0 \). But then the exact sequence (4) yields surjectivity of \( H^0(N^s) \to H^0(A) \), which is a contradiction unless \( A = 0 \).

Note, parenthetically, that there is no reason to think sections to \( N^s \) must go to zero in \( T_p \); thus we cannot apply the Vanishing Lemma directly to the kernel of the map \( N^s \to T_p \).

It remains to prove assertion (iii). To this end consider the exact sequence
\[
0 \to N^s(-1) \to N_L(-1) \to \bigoplus_{p \in Z} Q_p = 0 \tag{5}
\]
where \( Q_p \) is a skyscraper \( k(p) \)-module of length \( c - \lambda - 1 \) such that, writing
\[
\mathbb{P}^m = \mathbb{P}(V^*), L = \mathbb{P}(U^*), U \subseteq V,
\]
we may identify
\[
H^0(Q_p) = V/(U + T_p(X)(-1)).
\]
By our Vanishing Lemma, we have \( H^1(N^s(-1)) = 0 \), whence surjectivity of
\[
q : H^0(N_L(-1)) = \text{Hom}(U, V/U) \to \bigoplus H^0(Q_p).
\]
On the other hand, as \( N_L(-1) \) is a trivial bundle on \( L \simeq \mathbb{P}^{\lambda+1} \), it is easy to see from (5) that \( H^i(N^s(-1)) = 0, i > 1 \), and consequently
\[
\chi(N^s(-1)) = h^0(N^s(-1)) = (m - \lambda - 1) - k(c - \lambda - 1). \tag{6}
\]
Let \( W_i \) be the \( (m - c) \)-dimensional, \( (c - \lambda - 1) \)-codimensional subspace of \( V/U \) corresponding to \( T_p \). Note that each \( W_i \) projects to the tangent space of the branch of \( \pi_\Lambda(X) \) corresponding to \( p_i \). By (5), \( \bigcap W_i = H^0(N^s(-1)) \). Therefore (6) proves our transversality assertion. \( \square \)

It has been noted above that \( H^i(N^s(-i)) = 0, \forall i > 1 \), as follows from (5). Together with our Vanishing Lemma and the Castelnuovo-Mumford Lemma (\cite{6}, p. 99), we conclude

**Corollary 4.** The sheaf \( N^s \) is 0-regular. In particular, \( N^s \) is globally generated. \( \square \)

**Remark 5.** The general fibre of given length (e.g. 3) of generic projection from a line need not be in general position relative to the plane containing it. The following is an example of a smooth nondegenerate surface \( X \) in \( \mathbb{P}^5 \) admitting a 3-parameter family \( \{D_t\} \) of trisecant lines filling a hypersurface. Then, a generic line \( \Lambda \) will meet finitely many of the \( D_t \) and the corresponding aligned triples \( D_t \cap X \) will be fibres of projection from \( \Lambda \). Our vague impression- unbacked by evidence- is that such examples are not easy
to construct and may be ‘special’ in some characterizable way related to the defining equations.

Example 6. Let $F = (f_{ij})$ be a general symmetric $4 \times 4$ matrix of linear forms on $\mathbb{P}^5$. Let $X$ be the locus $\text{rk}(F) < 3$, i.e. the zero-locus of the $3 \times 3$ minors of $F$, which is a smooth surface cut out by cubics; let $Y$ be the locus $\text{rk}(F) < 4$, i.e. the zero-locus of $\det(F)$, which is a quartic singular along $X$. More invariantly, $F$ corresponds to a quadratic form $q$ on the trivial bundle $4\mathcal{O}_{\mathbb{P}^5}$ with values in $\mathcal{O}_{\mathbb{P}^5}(1)$, and $Y$ and $X$ are respectively the loci where $q$ drops rank by 1 or 2 at least. For $t = (t_1, ..., t_4)$ general, let $D_t$ be the line in $\mathbb{P}^5$ with equations $\sum_i t_i f_{ij}, j = 1, ..., 4$. The lines $D_t$ sweep out $Y$. I claim that a general $D_t$ is trisecant to $X$. To see this, we can assume by a suitable change of notations that $D_t$ corresponds to the 1st row (= 1st column) of $F$. Then the only $3 \times 3$ minor nontrivial on $D_t$ is the one corresponds to rows and columns 2, 3, 4, which gives a single cubic equation on $D_t$ defining the intersection $D_t \cap X$. □

Open questions. In conclusion, we point out 2 important questions that remain open.

(i) Let $Z$ be a general fibre of given length in a generic projection of a smooth variety. Then is $Z$ smoothable? (The question only arises if $Z$ has embedding dimension $> 2$.)

(ii) Describe the singularity of $X_k^L$ at a point $L$ such that $L \cap X$ has length $> k$ and embedding dimension $2$ (the case of embedding dimension $\leq 1$ was settled in [8]).

REFERENCES

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