A NOTE ON COTANGENT SHEAVES OF HILBERT SCHEMES OF FAMILIES OF PLANAR CURVES

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ABSTRACT. For a family of locally planar curves, the cotangent complex of the Hilbert scheme of points is equivalent to the Fourier-Mukai transform of the cotangent complex of the original family.

Hilbert schemes of zero-dimensional subschemes of locally planar curves and their deformation have been of interest recently, partly in connection with the work of Kool, Shende and Thomas [5] on Goettsche-type formulas for counting nodal curves on surfaces. Shende [12] subsequently showed that for a sufficiently general smoothing family $X/B$ with locally planar fibres, the relative Hilbert scheme $X_B^{[m]}$ is smooth for $m \leq \dim(B)$.

On the other hand, a tool much used in recent years to study various parameter and moduli spaces associated with a given variety $X$ is the so-called ‘Fourier-Mukai transform’ which relates sheaves and complexes on $X$ to sheaves and complexes on the parameter or moduli space. One important example of a complex associated to a scheme is the cotangent complex, so it is a natural question whether one can describe the cotangent complex of a parameter space, e.g. a Hilbert scheme associated to $X$ in terms of suitable objects on $X$, possibly via a Fourier-Mukai transform.

In what follows we will fix a family $\pi : X \to B$ of reduced, locally planar, not necessarily nodal, curves over $C$. Our purpose is to obtain a canonical formula for various cotangent sheaves and complexes associated with the relative Hilbert schemes $X_B^{[m]}$. In particular, we will extend to this setting a formula of Mattuck [9] for the cotangent sheaf of a symmetric product. Our formula (see Theorem 9) is a very natural one and in essence says that the cotangent complex of the Hilbert scheme is precisely the Fourier-Mukai transform of the cotangent complex of the curve or the family. See [7] or [4] [11] for more information on cotangent complexes in general.

The Fourier-Mukai functor in question is as usual associated to the tautological subscheme

$G \subset X \times X^{[m]}$

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An important role in what follows is played by the tautological rank-$m$ vector bundle $\Lambda_m(\mathcal{O}_X)$ on the Hilbert scheme $X^{[m]}$ and various subschemes thereof. The fibre of $\Lambda_m(\mathcal{O}_X)$ at a point $[I] \in X^{[m]}$ is $\mathcal{O}_X/I$. The sheaf $\Lambda_m(\mathcal{O}_X)$ is endowed with an inclusion 
$$\mathcal{O}_{X^{[m]}} \to \Lambda_m(\mathcal{O}_X),$$

fibrewise $\mathbb{C}1 \to \mathcal{O}_X/I$, which is split by the trace map
$$\text{tr} : \Lambda_m(\mathcal{O}_X) \to \mathcal{O}_{X^{[m]}}.$$

We will denote the kernel of $\text{tr}$, i.e. $\Lambda_m(\mathcal{O}_X)/\mathcal{O}_{X^{[m]}}$, by $\Lambda^0_m(\mathcal{O}_X)$; similarly, the image of a local section $t$ of $\mathcal{O}_X$ in $\Lambda^0_m(\mathcal{O}_X)$ will be denoted by $\Lambda^0_m(t)$. Thus, $\Lambda^0_m(t)$ vanishes at $[I]$ iff $t$ is constant on the subscheme $\text{Spec}(\mathcal{O}_X/I)$ of $X$ corresponding to $I$.

Given a map $\pi : X \to B$, we set $\Lambda_m(.) \otimes \mathcal{O}_{X^{[m]}} =: \Lambda_m(.)_B$. Note that given any quasi-coherent $B$-module $F$, we have
$$\Lambda_m(\pi^*(F))_B = (\pi^{[m]})^*(F) \otimes \Lambda_m(\mathcal{O}_X),$$

where $\pi^{[m]} : X_B^{[m]} \to B$ is the natural map. Moreover the trace map base-extends to a direct summand projection $\Lambda_m(\pi^*(F))_B \to (\pi^{[m]})^*(F)$, whose kernel will be denoted by $\Lambda^0_m(F)_B$.

A curve $X_0$ is said to be planar if it is reduced and has embedding dimension at most 2 at each point; equivalently, $X_0$ is locally embeddable in a smooth surface. A basic property of Hilbert schemes of planar curves $X_0$ is the following: let $p \in X_0$. Then in a smooth surface $S$ containing the germ of $X_0$ at $p$, the locus of length-$m$ schemes supported at $p$ is $(m-1)$-dimensional, by Briançon’s theorem [2]; therefore the dimension of the locus of length-$m$ subschemes of $X_0$ supported at $p$ is at most $(m-1)$-dimensional. This implies, by an easy dimension count, that any sublocus of $X_0^{[m]}$ consisting of schemes meeting the singular locus nontrivially must have dimension $< m$. On the other hand, if $t$ denotes a local defining equation for $X_0$ on $S$, then $X_0^{[m]}$ coincides with the zero-scheme of the canonical section $\Lambda_m(t)$ of the rank-$m$ bundle $\Lambda_m(\mathcal{O}_S(X_0))$ on the smooth $2m$-fold $S^{[m]}$, hence has no component of dimension $< m$. Thus:

**Lemma 1.** If $X_0$ is planar, then every component of the Hilbert scheme $X_0^{[m]}$ is $m$-dimensional and maps birationally via the cycle map onto a component of the symmetric product $X_0^{(m)}$. \hfill \Box

**Lemma 2.** Suppose $X/B$ has planar fibres and smooth base and total space. Then:

(i) The full Hilbert scheme $X^{[m]}$ is smooth along $X_B^{[m]}$.

(ii) $X_B^{[m]}$ is a locally complete intersection in $X^{[m]}$.

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Proof. (i) follows from the fact that any scheme in $X_B^{[m]}$ has embedding dimension $\leq 2$ and the standard fact that such schemes have unobstructed embedded deformations in any smooth variety. To see the latter briefly (see also [6]), note that if $z$ is a finite subscheme of embedding dimension 2 in a smooth variety $X$, then $z$ is contained in a smooth affine complete intersection surface $Y \subset X$. An infinitesimal deformation of $z$ in $X$ comes from a deformation of $Y$ which is unobstructed and trivializable ($Y$ being smooth and affine), so this deformation of $z$ in $X$ corresponds to a deformation of $z$ in $Y$, which is unobstructed by Fogarty’s theorem [1].

(ii) Set $d = \dim(B)$. Then $X_B^{[m]}$ has codimension $d(m-1)$ in $X^{[m]}$ by Lemma 1. On the other hand, working locally, if $t_1, ..., t_d$ is a local coordinate system on $B$, viewed as functions on $X$ they yield sections $\Lambda_m(t_i)$ of $\Lambda_m(O_X)$ whose common zero-locus is precisely the locus of subschemes $O_X/I$ on which $t_1, ..., t_d$ are constant, i.e. which are contained in a fibre of $X/B$. Thus, $X_B^{[m]}$ is locally defined by $d(m-1)$ equations on $X^{[m]}$.

Lemma 3. For any planar curve $X_0$, the Hilbert scheme $X_0^{[m]}$ is a locally complete intersection variety.

Proof. Working locally, choose a smoothing $X/B$, i.e. $X, B$ are smooth, $X/B$ is flat and has $X_0$ as fibre over $0 \in B$. Then as in the proof of Lemma 2, $X_0^{[m]}$ is the zero-scheme, on the smooth $m(d+1)$-dimensional variety $X^{[m]}$, of the $d$ sections $\Lambda_m(t_1), ..., \Lambda_m(t_d)$ of the rank-$m$ bundle $\Lambda_m(O_X)$, where $t_1, ..., t_d$ are local coordinates on $B$. Therefore $X_0^{[m]}$ is a locally complete intersection scheme; moreover by lemma 1, $X_0^{[m]}$ is birational to a sum of components of the symmetric product $X_0^{[m]}$, hence it is clearly generically reduced. Therefore $X_0^{[m]}$ has no embedded components, i.e. a it is variety.

Remark 4. After this note was first posted, it was pointed out that the Lemma was well known and could be found, e.g. in [1] or [3].

Note that the tangent space to the full Hilbert scheme $X^{[m]}$ at any $I \in X_B^{[m]}$, is $\text{Hom}(I, O/I)$. The tangent space to $X_B^{[m]}$ is the subspace of $\text{Hom}(I, O/I)$ consisting of maps sending $m_{B,b} \subset I$ to $O_B/m_{B,b} = \mathcal{C}1 \subset O/I$, i.e. such that the composite $m_{B,b} \rightarrow (O/I)^\times$ vanishes, where $(O/I)^\times$ is the kernel of the trace map. This composite takes $m_{B,b}^2$ to $m_{B,b}(O/I)^\times = 0$ (as $O/I$ is an $O_{X_B}$-module), therefore the composite a priori factors through $m_{B,b}/m_{B,b}^2$. Thus we have an exact sequence

$$0 \rightarrow T_{X_B^{[m]}, I} \rightarrow T_{X^{[m]}, I} \rightarrow (m_{B,b}/m_{B,b}^2)^* \otimes (O/I)^\times.$$

Note that $(O/I)$ is self-dual under the trace pairing, and the subspace $\mathcal{C}1$ is not self-orthogonal, since the characteristic is zero. Therefore the orthogonal complement to $\mathcal{C}1$,
i.e. \((\mathcal{O}/I)^0\), inherits a nondegenerate trace pairing, so \((\mathcal{O}/I)^0\) is self-dual. Therefore dualizing the latter sequence, and globalizing over \(X_B^{[m]}\), we get exact

\[
\Lambda^0_m(\pi^*(\Omega_B))_B \to \Omega_{X_B^{[m]}} \otimes \mathcal{O}_{X_B^{[m]}} \to \Omega_{X_B^{[m]}} \to 0.
\]

Note that the left arrow above is generically injective, hence injective since its source is locally free and \(X_B^{[m]}\) is integral.

Similarly, the tangent space to \(X_0^{[m]}\) consists of maps \(I \to \mathcal{O}/I\) which kill \(m_B\) outright, and this leads to an analogous sequence without the \(^0\) superscript. Thus we conclude:

**Lemma 5.** For any flat family \(X \xrightarrow{\pi} B\) of planar curves with smooth base and total space, we have exact sequences:

\[\begin{align*}
0 & \to \Lambda^0_m(\pi^*(\Omega_B))_B \to \Omega_{X_B^{[m]}} \otimes \mathcal{O}_{X_B^{[m]}} \to \Omega_{X_B^{[m]}} \to 0 \\
0 & \to \Lambda_m(\pi^*(\Omega_B))_B \to \Omega_{X_0^{[m]}} \otimes \mathcal{O}_{X_0^{[m]}} \to \Omega_{X_0^{[m]}} / B \to 0
\end{align*}\]

As an immediate consequence, we obtain

**Corollary 6.** Let \(X_0\) be a planar curve and \(\pi : X \to B\) a smoothing of \(X_0\) over a smooth base. Then

(i) the conormal to \(X_B^{[m]}\) in \(X^{[m]}\) is \(\Lambda^0_m(\pi^*(\Omega_B))_B\);

(ii) the conormal to \(X_0^{[m]}\) in \(X^{[m]}\) is \(\Lambda_m(\pi^*(\Omega_B))_B\).

**Proof.** The first assertion follows directly from (1). The second follows similarly from (2) by tensoring with \(\mathcal{O}_{X_0^{[m]}}\) and noting that the map \(\Lambda_m(\mathcal{O}_X) \otimes (\pi^{[m]})^*(\Omega_B) \otimes \mathcal{O}_{X_0^{[m]}} \to \Omega_{X_0^{[m]}} \otimes \mathcal{O}_{X_0^{[m]}}\) is still generically injective, hence injective. \(\square\)

**Lemma 7.** For \(X/B\) has planar fibres and smooth total space, we have an isomorphism

\[\text{tr} : \Lambda_m(\Omega_X)_B \to \Omega_{X_B^{[m]}} \otimes \mathcal{O}_{X_B^{[m]}}\]

**Proof.** The map is the trace map for differentials. If \(U \subset X\) denotes the complement of the fibre singularities, it is easy to see, and already follows from Mattuck [9], that the map is surjective over \(U_B^{[m]}\). By Briançon’s theorem, the complement of \(U_B^{[m]}\) has codimension > 1. Hence, being a map of bundles of the same rank, \(\text{tr}\) is an isomorphism. \(\square\)

**Remark 8.** Mattuck’s result is the case where \(B\) is a point.

Now suppose that \(X/B\) is a smoothing of a planar curve \(X_0\), and consider the universal subscheme \(D^{m,1} \subset X_B^{[m]} \times_B X\). Because \(X_B^{[m]}\) is a locally complete intersection, it is Cohen-Macaulay. Because \(D^{m,1}\) is flat over \(X_B^{[m]}\), a regular sequence for \(X_B^{[m]}\) lifts to one for \(D^{m,1}\), and by finiteness it follows that \(D^{m,1}\) is Cohen-Macaulay. By Lemma [7], the fibres of \(D^{m,1}\) over \(X\) are all \((m-1)\)-dimensional. Therefore \(D^{m,1}\) is flat over \(X\) (see [8], p.
It follows, firstly that the functor $\Lambda_m$ is exact, hence extends termwise to the derived category (it is an analogue of the well-known Fourier-Mukai transform). Further, by smoothness of $X$, we can identify the cotangent complex $L_{X_0} = L_{X_0/\mathbb{C}}$ of a fibre $X_0$ with $[\pi^*(\Omega_B) \otimes \mathcal{O}_{X_0} \to \Omega_X \otimes \mathcal{O}_{X_0}]$ and likewise $L_{X/B} \sim [\pi^*(\Omega_B) \to \Omega_X] \sim \Omega_{X/B}$. Similarly, $X_0^{[m]}$ is a fibre of $X_B^{[m]}$ with smooth total space, hence the cotangent complex of $X_0^{[m]}$ can be identified:

$$L_{X_0^{[m]}} \sim [\Lambda_m(\pi^*(\Omega_B)) \otimes \mathcal{O}_{X_0^{[m]}} \to \Omega_{X^{[m]}} \otimes \mathcal{O}_{X_0^{[m]}}].$$

Likewise, the relative cotangent complex of $X_B^{[m]}/B$ can be identified:

$$L_{X_B^{[m]}/B} \sim [\Lambda_m(\pi^*(\Omega_B)) \to \Omega_{X^{[m]}} \otimes \mathcal{O}_{X_B^{[m]}}].$$

Now using Lemma 2 to identify the second term in the two complexes above, we conclude

**Theorem 9.** (i) Let $X_0$ be a planar curve. Then the absolute cotangent complex $L_{X_0^{[m]}/\mathbb{C}}$ is equivalent to $\Lambda_m(L_{X_0/\mathbb{C}})$.

(ii) Let $X/B$ have planar fibres and smooth total space and base. Then the relative cotangent complex $L_{X_B^{[m]}/B}$ is equivalent to $\Lambda_m(L_{X/B})$.

**Remark 10.** If $X/B$ is a sufficiently general smoothing of an irreducible planar curve $X_0$, then Shende [12] has shown that $X_B^{[m]}$ is smooth for $m \leq \dim(B)$. In that case the sequence (1) is locally split; in general, $X_B^{[m]}$ is singular even if $X$ is smooth and $X_0$ has only ($> 1$) nodes, in which case (1) is not locally split.

Finally we consider an application to families of nodal curves $X/B$. See [10], especially §1, for details on the structure of the Hilbert scheme in this case. We denote by $\Gamma^{(m)}$ the discriminant (Cartier) divisor on $X_B^{[m]}$ and $\mathcal{O}_{X_B^{[m]}}(1) = \mathcal{O}(-\Gamma^{(m)})$. For any divisor $D$ on $X$, there is an associated divisor $[m]_*(D)$ on $X_B^{[m]}$, supported on the locus of schemes meeting the support of $D$. This construction extends to line bundles, and we have

$$\det(\Lambda_m(L)) = [m]_*(L) \otimes \mathcal{O}(1).$$

In particular,

$$\det(\Lambda_m(\mathcal{O}_X)) = \mathcal{O}(1).$$

**Corollary 11.** If $X/B$ is any flat family of nodal curves so that $B$ is a locally Gorenstein, we have

$$\mathcal{O}_{X_B^{[m]}}(1) \simeq \omega_{X_B^{[m]}} \otimes [m]_*(\omega_X^{-1})$$

$$\simeq \omega_{X_B^{[m]}/B} \otimes [m]_*(\omega_{X/B}^{-1}).$$

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Proof. Note that all three line bundles in question are compatible with base-change and the last 2 are clearly isomorphic as $\omega_B$ is locally free. Therefore we may assume $X, B$ are smooth and $X/B$ is versal. Consider the map of coherent sheaves coming from the trace map

$$\Lambda_m(\Omega_{X/B})_B \to \Omega_{X_B^{[m]}/B}$$

(4)

If $U \subset X$ denotes as above the set of smooth points of fibres, whose complement has codimension 2, we get a map defined on $U_B^{[m]}$

$$\Lambda_m(\omega_{X/B})_B \to \Omega_{X_B^{[m]}/B}.$$ 

Taking determinants, we get a map over $U_B^{[m]}$

$$\bigodot([m]_*(\omega_{X/B} - \Gamma)) \to \omega_{X_B^{[m]}/B}.$$ 

It is easy to check (as was done by Mattuck [9]) that this is an isomorphism over $U_B^{[m]}$. Since $U_B^{[m]}$ is the complement of a codimension-2 locus, the isomorphism extends globally. Therefore the versions without the $/ B$ are also isomorphic. □
REFERENCES

2. J. Briançon, *Description de Hilb\(^n\)C\{x,y\},* Inventiones math. 41 (1977), 45–89.