

# Representations of Braid Groups and Colored Homfly Polynomials

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- In the abstract of his seminal paper “Hecke algebra representations of braid groups and link polynomials” (Ann. of Math. 1987), Vaughan Jones wrote: “*By studying representations of the braid group satisfying a certain quadratic relation we obtain a polynomial invariant in two variables for oriented links. ... The two-variable polynomial was first discovered by Freyd-Yetter, Lickorish-Millet, Ocneanu, Hoste, and Przytycki-Traczyk.*”

- Since then, this two-variable link polynomial has been generalized to the notion of colored  $U_q(\mathfrak{sl}_N)$  quantum link polynomial with two variables  $q$  and  $q^N$ . We will refer to this generalization as a *colored homfly polynomial*.

- Despite of the fact that this is now a well developed theory, the computation of colored homfly polynomials is still extremely challenging. A general formula seems to exist in literature only for the Hopf link (Morton-Lukac, JKTR 2003).
- Yet, a detailed understanding of the colored homfly polynomial is extremely important, for its relationships with hyperbolic geometry as well as string theory. Some of the references about the conjectured relationship between the large  $N$  asymptotics of the colored homfly polynomial with the Gromov-Witten invariants of certain noncompact Calabi-Yau 3-folds will be given in the end of the talk.

## A representation of braid groups

- $V$  = vector space with a basis  $\{X_1, \dots, X_N\}$ .
- $V^{\otimes n}$  = the tensor power of  $V$ , with the basis elements  $X_{(j_1 j_2 \dots j_n)} = X_{j_1} \otimes X_{j_2} \otimes \dots \otimes X_{j_n}$ .
- $B_n$  = braid group of  $n$  strands, with the standard generators  $\sigma_1, \dots, \sigma_{n-1}$ .

- A representation  $\rho : B_n \rightarrow \text{End}_{\mathbb{C}}(V^{\otimes n})$  can be defined as follows:

$$\rho(\sigma_i) \left( X_{(\dots j_i j_{i+1} \dots)} \right) = X_{(\dots j_{i+1} j_i \dots)}$$

when  $j_i \leq j_{i+1}$ , and

$$= (1 - t) X_{(\dots j_i j_{i+1} \dots)} + t X_{(\dots j_{i+1} j_i \dots)}$$

when  $j_i \geq j_{i+1}$ .

- This representation  $\rho$  of the braid group  $B_n$  is simple yet fundamental. It contains almost all information of the representation theory of Hecke algebras and that of the quantum group  $U_q(\mathfrak{sl}_N)$ .
- This representation also leads to a *deformation* of the entire representation theory of symmetric groups, especially the combinatorial aspects of it.
- For a positive braid  $\beta$ ,  $\rho(\beta)$  gives rise to a Markov process. The dynamical properties of this Markov process are intimately related with the geometric properties of the braid  $\beta$ .

• Consider  $V$  as the vector representation of the quantum group  $U_q(\mathfrak{sl}_N)$ , we have a representation  $h_V : B_n \rightarrow \text{End}_{U_q(\mathfrak{sl}_N)}(V^{\otimes n})$ . Let

$$\widetilde{h}_V(\sigma_i) = q^{(1-N)/N} h_V(\sigma_i).$$

**Lemma.** *Let  $t = q^{-2}$ . Under the change of bases:*

$$X_{(j_1 j_2 \dots j_n)} \mapsto q^{\#\{(k,l); k < l, j_k < j_l\}} X_{(j_1 j_2 \dots j_n)},$$

*the braid group representations  $\rho$  and  $\widetilde{h}_V$  are equivalent to each other. In particular, they both satisfy the Hecke algebra relation*

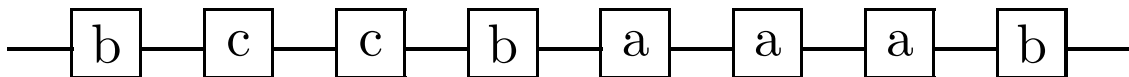
$$(g_i - 1)(g_i + t) = 0.$$

• For a composition  $\mu = (\mu_1, \mu_2, \dots, \mu_N)$  of  $n$ , we have a subspace  $M^\mu$  of  $V^{\otimes n}$  spanned by  $X_{j_1} \otimes X_{j_2} \otimes \dots \otimes X_{j_n}$  where each  $X_i$  appears exactly  $\mu_i$  times. Each  $M^\mu$  has a basis whose elements are in one-on correspondence with Young tabloids of the shape  $\mu$ . And each  $M^\mu$  is a  $B_n$  module.

**Example:**  $M^{(n-1,1)}$  is precisely the Burau representation of  $B_n$ .

- For each  $\mu$ , a positive braid in  $B_n$  gives rise to a model of random walks on Young tabloids of the shape  $\mu$ .

**Example:**



This is a Young tabloid of the shape  $\mu = (3, 3, 2)$ . Other Young tabloids of the same shape are obtained by filling in these 8 boxes with three a's, three b's, and two c's, one letter for each box.

## The homfly polynomial

- For  $h \in \text{End}_{U_q(\mathfrak{sl}_N)}(V)$ , if  $h(X_i) = \sum h_{ij} X_j$ , we have the quantum trace:

$$\text{tr}_q(h) = \sum_i h_{ii} q^{N+1-2i} = \sum_i h_{ii} t^{(2i-N-1)/2}.$$

Let  $\beta \in B_n$ . The homfly polynomial of the link  $\hat{\beta}$  is

$$P_N(\hat{\beta}) = t^{e(\beta)(N-1)/2} \left( \frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{t^{\frac{N}{2}} - t^{-\frac{N}{2}}} \right) \text{tr}_q(\rho(\beta)).$$

The homfly polynomial  $P_N(\hat{\beta})$  is characterized by the skein relation:

$$t^{\frac{N}{2}} P_N(L_+) - t^{-\frac{N}{2}} P_N(L_-) = \left( t^{\frac{1}{2}} - t^{-\frac{1}{2}} \right) P_N(L_0).$$

- To calculate the quantum trace  $\text{tr}_q(\rho(\beta))$ , we consider the decomposition

$$V^{\otimes n} = \bigoplus_{\mu_1 + \mu_2 + \dots + \mu_N = n} M^{(\mu_1, \mu_2, \dots, \mu_N)}.$$

Each permutation module  $M^\mu$  can be further decomposed into irreducible Specht modules  $S^\lambda$ , one for each partition  $\lambda$  of  $n$ :

$$M^\mu \cong \bigoplus_{\lambda \vdash n} K_{\lambda\mu} S^\lambda,$$

where  $K_{\lambda\mu}$  is the multiplicity (Kostka number).

- $\xi^\mu =$  the character of the  $B_n$ -module  $M^\mu$ , for  $\mu = (\mu_1, \mu_2, \dots, \mu_N) \models n$ .
- $\zeta^\lambda =$  the character of the  $B_n$ -module  $S^\lambda$ , for  $\lambda \vdash n$ .

**Theorem A.** (Ocneanu, Wenzl) *The quantum trace  $\text{tr}_q(\beta)$  can be expressed as follows:*

$$\text{tr}_q(\beta) = \sum_{\lambda \vdash n} \zeta^\lambda(\beta) s_\lambda(t^{-(N-1)/2}, t^{-(N-3)/2}, \dots, t^{(N-1)/2}),$$

where  $s_\lambda(x_1, x_2, \dots, x_N)$  is the Schur function.

**Proof:** In each permutation module  $M^\mu$  of  $V^{\otimes n}$ , the weight of the quantum trace on basis elements is a constant equal to

$$\prod_{i=1}^N t^{\mu_i(2i-N-1)/2}.$$

Thus,

$$\mathrm{tr}_q(\beta) = \sum_{\mu \models n} \xi^\mu(\beta) \prod_{i=1}^N t^{\mu_i(2i-N-1)/2}.$$

Recall that the Schur function is defined as

$$s_\lambda(x_1, x_2, \dots, x_N) = \sum_{\mu \models n} K_{\lambda\mu} \prod_{i=1}^N x_i^{\mu_i}.$$

Therefore, we have the desired expression of the quantum trace. □

## Colored homfly polynomials

- A basic fact is that every irreducible  $U_q(\mathfrak{sl}_N)$  module sits inside of the tensor power  $V^{\otimes k}$ . For a braid  $\beta \in B_n$ , let  $\beta^{(k)}$  denote the  $k$ -cabling of  $\beta$ . Then we have

$$\rho\left(\beta^{(k)}\right) \in \text{End}_{U_q(\mathfrak{sl}_N)}\left((V^{\otimes kn})\right).$$

Use projections, we will find an expression for a colored homfly polynomial similar to the one given in Theorem A for the homfly polynomial.

- A composition  $\mu = (\mu_1, \dots, \mu_N)$  gives rise to a weight  $\widehat{\mu} = (\mu_1 - \mu_2, \mu_2 - \mu_3, \dots, \mu_{N-1} - \mu_N)$  of  $U_q(\mathfrak{sl}_N)$ .
- A partition  $\lambda$  of length  $N$  gives rise to an irreducible  $U_q(\mathfrak{sl}_N)$ -module  $U^{\widehat{\lambda}}$  with the highest weight  $\widehat{\lambda}$ . The irreducible  $U_q(\mathfrak{sl}_N)$ -modules that appear in  $V^{\otimes k}$  are in one-one correspondence to partitions  $\lambda \vdash k$  of length  $N$ .

- $p_\lambda \in \text{End}_{U_q(\mathfrak{sl}_N)}(V^{\otimes k})$  denotes the minimal projection with  $p_\lambda V^{\otimes k} \cong U^{\hat{\lambda}}$ .

- For a partition  $\lambda \vdash k$ , the quantum trace

$$P_{U^{\hat{\lambda}}}(\hat{\beta}) = \text{tr}_q(p_\lambda^{\otimes n} \beta^{(k)})$$

is a framed link invariant. It can be normalized to the framing independent link invariant  $P_{U^{\hat{\lambda}}}^0(\hat{\beta})$ . We will call this link invariant a colored homfly polynomial and denote it by  $J_\lambda(L)$ .

**Theorem B.** *The quantum trace can be expressed as follows:*

$$P_{U^{\hat{\lambda}}}(\hat{\beta}) = \text{tr}_q(p_\lambda^{\otimes n} \rho(\beta^{(k)})) = \sum_{\mu \vdash nk} \zeta^\mu(p_\lambda^{\otimes n} \beta^{(k)}) s_\mu(q^{N-1}, q^{N-3}, \dots, q^{-(N-1)}).$$

- Colored homfly polynomial  $J_\lambda(L)$  may be thought of as a function of the partition  $\lambda$ . The asymptotics of this function for large  $N = \text{length}(\lambda)$  is an uncharted territory in quantum topology.

**Theorem C.** *For every partition  $\lambda$  and a knot  $L$ , we have*

$$J_\lambda(L)/J_\lambda(\text{unknot}) \in \mathbb{Z}[q^{\pm 2}].$$

- Note that the coefficients of the representation  $\rho$  on a positive braid are all nonnegative for  $q > 1$ . Based on this observation, some positivity results about  $J_\lambda(L)$  can be obtained when  $L$  is a positive closed braid.

## The link invariant of $\widehat{\Delta}_n^{2m}$

- Let  $\Delta_n$  be the half twist of the trivial  $n$ -strand braid. Then  $\widehat{\Delta}_n^{2m}$  is a link of  $n$  components. The Hopf link  $\widehat{\Delta}_2^2$  is a member of this family of links.
- Associated with a collection of  $n$  partitions  $\lambda^i \vdash k_i$ ,  $i = 1, 2, \dots, n$ , of length  $N$ , we have the colored homfly polynomial

$$P_{U^{\widehat{\lambda}^1}, \dots, U^{\widehat{\lambda}^n}}^0(\widehat{\Delta}_n^{2m}),$$

where the  $i$ -th component of the link is colored by the irreducible  $U_q(\mathfrak{sl}_N)$ -module  $U^{\widehat{\lambda}^i}$ . We shall be able to calculate this link polynomial in terms of the combinatorial data involved. For simplicity, we will only give the result for the quantum trace

$$\mathrm{tr}_q \left( (p_{\lambda^1} \otimes \cdots \otimes p_{\lambda^n}) (\Delta_{k_1}^{-2m} \cdots \Delta_{k_n}^{-2m} \Delta_k^{2m}) \right),$$

where  $k = k_1 + \cdots + k_n$ .

*Combinatorial data:*

- For a partition  $\lambda = (\lambda_1, \lambda_2, \dots)$ , and its conjugation  $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ ,

$$\kappa_\lambda = \sum_i \sum_{j=1}^{\lambda_i} 2(j-i) = \sum_i \lambda_i^2 - \sum_i \lambda'_i{}^2.$$

- We have the decomposition

$$U^{\widehat{\lambda}^1} \otimes \dots \otimes U^{\widehat{\lambda}^n} \cong \bigoplus_{\lambda \vdash k} d_\lambda U^{\widehat{\lambda}}.$$

The multiplicity  $d_\lambda$  can be calculated as follows:

$$d_\lambda = \sum_{\mu^1, \dots, \mu^{n-2}} c_{\lambda^1 \lambda^2}^{\mu^1} c_{\mu^1 \lambda^3}^{\mu^2} \cdots c_{\mu^{n-3} \lambda^{n-1}}^{\mu^{n-2}} c_{\mu^{n-2} \lambda^n}^{\lambda}$$

where  $c_{\mu\nu}^\lambda$ 's are the Littlewood-Richardson coefficients.

**Theorem D.** *We have the following expression for the colored quantum trace of  $\Delta_n^{2m}$ :*

$$\begin{aligned}
& P_{U^{\widehat{\lambda^1}}, \dots, U^{\widehat{\lambda^n}}}(\widehat{\Delta_n^{2m}}) \\
&= \text{tr}_q \left( (p_{\lambda^1} \otimes \dots \otimes p_{\lambda^n}) (\Delta_{k_1}^{-2m} \dots \Delta_{k_n}^{-2m} \Delta_k^{2m}) \right) \\
&= \left( q^{2 \sum_{i < j} k_i k_j} \right)^{-\frac{m}{N}}. \\
& \sum_{\lambda \vdash k} d_\lambda \left( q^{\kappa_\lambda - \sum_i \kappa_{\lambda i}} \right)^m s_\lambda \left( q^{N-1}, q^{N-3}, \dots, q^{-(N-1)} \right).
\end{aligned}$$

Note that in this expression, the dependence of  $N$  is only via the Schur functions and the exponent  $-m/N$  of the first terms.

- The colored quantum trace of  $\Delta_n^{2m}$  can also be calculated inductively.

**Theorem E.** *With*

$$\begin{aligned} P_{U^{\widehat{\lambda}}}(\text{unknot}) &= \dim_q U^{\widehat{\lambda}} \\ &= s_{\lambda}(q^{N-1}, q^{N-3}, \dots, q^{-(N-1)}), \end{aligned}$$

*we have the recursion formula*

$$\begin{aligned} &P_{U^{\widehat{\lambda}^1}, \dots, U^{\widehat{\lambda}^n}}(\widehat{\Delta_n^{2m}}) \\ &= (q^{2k_1 k_2})^{-\frac{m}{N}}. \\ &\sum_{\mu} c_{\lambda^1 \lambda^2}^{\mu} (q^{\kappa_{\mu} - \kappa_{\lambda^1} - \kappa_{\lambda^2}})^m P_{U^{\widehat{\mu}}, U^{\widehat{\lambda}^3}, \dots, U^{\widehat{\lambda}^n}}(\widehat{\Delta_{n-1}^{2m}}). \end{aligned}$$

• These calculations are of potential importance for the study of the relationship between Chern-Simons gauge theory and topological string theory on noncompact Calabi-Yau spaces. See the references below:

[1] Marcos Marino and Cumrun Vafa, *Framed knots at large  $N$* .

[2] J. M. F. Labastida, Marcos Marino, and Cumrun Vafa, *Knots, links and branes at large  $N$* .

[3] Mina Aganagic, Albrecht Klemm, Marcos Marino, and Cumrun Vafa, *The Topological vertex*.

[4] Chiu-Chu Melissa Liu, Kefeng Liu, and Jian Zhou, *A Proof of a conjecture of Marino-Vafa on Hodge integrals*.

[5] Chiu-Chu Melissa Liu, Kefeng Liu, and Jian Zhou, *A formula for two partition Hodge integrals*.

[6] Jun Li, Chiu-Chu Melissa Liu, Kefeng Liu, and Jian Zhou, *A Mathematical theory of the topological vertex*.