Generalised filters 1

M.H. Burton\textsuperscript{a,*}, M. Muraleetharan\textsuperscript{a}, J. Gutiérrez García\textsuperscript{b}

\textsuperscript{a} Department of Mathematics, Rhodes University, Grahamstown, 6140, South Africa
\textsuperscript{b} Matematika Saila, Euskal Herriko Unibertsitatea, 48080, Bilbo, Spain

Received March 1997; received in revised form August 1997

Abstract

The notion of a filter is generalised and the resulting mathematical object is called a generalised filter. These filters fit naturally into the context of generalised uniform spaces. The basic theory of generalised filters is established and the relationship between generalised filters and prefilters is discovered. © 1999 Elsevier Science B.V. All rights reserved.

\textit{AMS classification:} 03E72; 04A20; 04A72; 18A22; 54A40; 54E15

\textit{Keywords:} Filter; Ultrafilter; Prefilter; Prime prefilter; Saturated prefilter; Fuzzy filter; Uniformity; Fuzzy uniformity; Super uniformity

1. Introduction

Uniform spaces were introduced to form an appropriate abstract setting for the study of completeness. These spaces, which fall between metric spaces and topological spaces, have been extensively studied and the reader is referred to [25] for the basic theory of uniform spaces. The study of uniform space notions is facilitated by the notion of a filter and the basic theory of filters can also be found in [25]. For the record, let us just note that a uniformity on a set \( X \) is a subset \( D \subseteq 2^{X \times X} \) and a filter on \( X \) is a subset \( F \subseteq 2^X \).

In other words
\[
D \in 2^{2^{X \times X}}, \quad F \in 2^X.
\]

Since the notion of a fuzzy set was introduced by Zadeh [28], there has been an attempt to extend useful mathematical notions to this wider setting, replacing sets by fuzzy sets. A fuzzy set, \( \mu \), on a set \( X \) is simply a function \( \mu : X \rightarrow I \), where \( I \) denotes the unit interval. In [20], Lowen introduced the notion of a fuzzy uniform space and the fuzzy uniform space analogues of compactness, relative compactness, completeness, precompactness and boundedness have now been extended to the fuzzy setting [1–5, 9, 10, 21, 22]. The basic theory of function spaces has also been extended to this setting [6]. This was accomplished using the theory of prefilters [18, 19]. The reader is referred to [2], Section 2, [15], Section 2 for a summary of the basic theory of prefilters. Let us recall that a fuzzy uniformity on a set \( X \) is a subset \( D \subseteq I^{X \times X} \) and a prefilter on \( X \) is a subset \( \mathcal{F} \subseteq I^X \).

In other words
\[
D \in 2^{I^{X \times X}}, \quad \mathcal{F} \in 2^I.
\]

The reader is referred to [2, Section 3], for a summary of the basic theory of fuzzy uniform spaces.
Recently, in [14], the notion of a uniform space has been generalised even further. A super uniformity, $\delta$, on a set $X$ is defined to be a function

$$\delta : 2^X \times 2^X \rightarrow I,$$

satisfying certain conditions. In other words,

$$\delta \in I^{2^X \times 2^X}.$$

As might be expected, a fundamental tool for investigating super uniform spaces is the analogue of a filter in this setting, which is called a fuzzy filter. A fuzzy filter, $\varphi$, on $X$ is a function $\varphi : 2^X \rightarrow I$ satisfying certain conditions.

In other words,

$$\varphi \in I^X.$$

In [7], another extension of the notion of a uniform space was defined and investigated. A generalised uniformity, $\sigma$, on a set $X$ is defined to be a function $\sigma : 2^{X \times X} \rightarrow I$ satisfying certain natural conditions. In other words,

$$\sigma \in I^{2^{X \times X}}.$$

There are natural injections between the sets as indicated

$$\begin{array}{c}
I^{2^X} \leftarrow \downarrow \leftarrow \downarrow \leftarrow \downarrow \\
\downarrow \downarrow \downarrow \downarrow \\
2^X \times 2^X \leftarrow \downarrow \leftarrow \downarrow \leftarrow \downarrow \\
\downarrow \downarrow \downarrow \downarrow \\
2^{X \times X} \leftarrow \downarrow \leftarrow \downarrow \leftarrow \downarrow \\
\downarrow \downarrow \downarrow \downarrow \\
I^{2^{X \times X}}
\end{array}$$

and it is shown in [15] that there are categorical embeddings between the categories: $US$ (of uniform spaces), $FUS$ (of fuzzy uniform spaces), $GUS$ (of generalised uniform spaces) and $SUS$ (of super uniform spaces) as indicated

$$\begin{array}{c}
SUS \leftarrow \downarrow \leftarrow \downarrow \leftarrow \downarrow \\
\downarrow \downarrow \downarrow \downarrow \\
GUS \leftarrow \downarrow \leftarrow \downarrow \leftarrow \downarrow \\
\downarrow \downarrow \downarrow \downarrow \\
FUS \leftarrow \downarrow \leftarrow \downarrow \leftarrow \downarrow \\
\downarrow \downarrow \downarrow \downarrow \\
US
\end{array}$$

Although prefilters were used in the investigation of generalised uniform spaces, it would seem that, in this context, the natural analogue of a filter on $X$ would be a function $f : 2^X \rightarrow I$. In other words, an element

$$f \in I^X,$$

satisfying certain conditions. We show that such an analogue, which we shall call a generalised filter, does exist and we establish the basic theory of these generalised filters. We first recall some relevant definitions, notation and results.

2. Preliminaries

For the sake of fixing notation, we recall some basic definitions.

We shall let $I$ denote the closed unit interval $[0, 1]$

and we let $I_0 \overset{\text{def}}{=} I \setminus \{0\} = (0, 1]$, $I_1 \overset{\text{def}}{=} I \setminus \{1\} = [0, 1)$. We denote the characteristic function of a subset $A \subseteq X$ by $1_A$. If $\mu \in I^X$ we define

$$\mu^x = \{x \in X : \mu(x) > x\}.$$

If $X$ is a set then a filter on $X$ is a nonempty subset $F \subseteq 2^X$ satisfying the following conditions:

F1 \ $\emptyset \notin F$;
F2 \ $F, G \in F \Rightarrow F \cap G \in F$;
F3 \ $F \in F$, $F \subseteq G \Rightarrow G \in F$.

If $F$ is nonempty and satisfies

FB1 \ $\emptyset \notin F$;
FB2 \ $\forall F, G \in F, \ \exists H \in F, \ H \subseteq F \cap G$

we call $F$ a filter base on $X$.

A prefilter on $X$ is a nonempty subset $\mathcal{F} \subseteq I^X$ which satisfies the following conditions:

PF1 \ $0 \notin \mathcal{F}$;
PF2 \ $\forall \nu, \mu \in \mathcal{F} \Rightarrow \nu \wedge \mu \in \mathcal{F}$;
PF3 \ $\exists \nu \in \mathcal{F}, \ \forall \mu \in \mathcal{F} \Rightarrow \mu \in \mathcal{F}$.

If $\mathcal{F}$ is nonempty and satisfies

PFB1 \ $0 \notin \mathcal{F}$;
PFB2 \ $\forall v_1, v_2 \in \mathcal{F}, \ \exists v_3 \in \mathcal{F}, \ v_3 \leq v_1 \wedge v_2$

we call $\mathcal{F}$ a prefilter base on $X$.

Naturally, a filter is a filter base and a prefilter is a prefilter base.

If $X$ is a set and $F$ is a family of subsets of $X$, we define

$$\langle F \rangle \overset{\text{def}}{=} \{A \subseteq X : \exists F \in F, \ F \subseteq A\}.$$
Similarly, if \( F \subseteq I^X \) we define
\[
\langle F \rangle \overset{\text{def}}{=} \{ \mu \in I^X : \exists v \in F, v \leq \mu \}. 
\]

If \( F \) is a prefilter base on \( X \) we define the characteristic, \( c(F) \), of \( F \) by
\[
c(F) = \inf \sup_{v \in F} v. 
\]

If \( F \) is a prefilter base and \( c(F) > 0 \), we define
\[
\hat{F} \overset{\text{def}}{=} \left\{ \sup_{\mu \in F}(\mu - \varepsilon) : (\mu_v) \in F_0 \right\}
\]
and call \( \hat{F} \) the saturation of \( F \). We say that \( F \) is saturated if \( \hat{F} = F \).

Note that in case \( c(F) = c = 0 \), if we define \( \hat{F} \) as before then we have \( 0 \in \hat{F} \). We conclude that if \( F \) is saturated then \( c(F) > 0 \).

If \( F \) is a prefilter base, \( c(F) = c > 0 \) and \( 0 < \alpha \leq c \) then we define
\[
F^\beta = \{ v \in F \mid \beta < \alpha \}.
\]

On the other hand, if \( F \) is a filter on \( X \) and \( \alpha \in (0, 1] \) then we define
\[
F^\alpha = \{ v \in I^X \mid \forall \beta < \alpha, v^\beta \in F \}.
\]

The reader is referred to [19, 20, 18] for the basic notation and theory of prefilters as well as to [2, 15]. As regards notation, we follow the notation established in [2] which is the notation of Lowen, with some modifications.

We list some of the main results here. Firstly, concerning prefilter bases:

**Theorem 2.1.** Let \( X \) be a set and \( F \) a prefilter base on \( X \), \( c(F) = c > 0, 0 < \alpha \leq c \). Then
1. \( \langle F \rangle \) is a prefilter and \( F \subseteq \langle F \rangle \);
2. \( c(F) = c(\langle F \rangle) \);
3. \( F^\beta \) is a filter base;
4. \( F^\alpha = \bigcup_{0 < \beta < \alpha} F^\beta \);
5. If \( c > 0 \) then \( \hat{F} \) is a prefilter base and \( F \subseteq \hat{F} \);
6. If \( c > 0 \) then \( c(F) = c(\hat{F}) \);
7. If \( c > 0 \) then \( \hat{F} = \langle \hat{F} \rangle = \langle \langle F \rangle \rangle \) is a saturated prefilter.

Secondly, concerning prefilters:

**Theorem 2.2.** Let \( X \) be a set and \( F \) a prefilter on \( X \), \( c(F) = c > 0, 0 < \alpha \leq c \). Then
1. \( \hat{F} = \hat{F} \) is a prefilter and \( F \subseteq \hat{F} \);
2. \( F^\alpha \) is a filter;
3. \( \hat{F} = \{ v \in I^X : \forall \varepsilon > 0, v + \varepsilon \in F \} \);
4. \( c(F) = \inf \{ \alpha \in I : \alpha X \in F \} = \sup \{ \alpha \in I : \alpha X \subseteq \hat{F} \} \).

The proofs of the results in the last two theorems can be found in the cited literature or supplied by the reader. The next result is crucial and we include a proof.

**Theorem 2.3.** Let \( F \) be a prefilter, \( c(F) = c > 0 \) and \( 0 < \alpha \leq c \). Then
\[
F^\alpha = (\hat{F})^\alpha.
\]

**Proof.** Since \( F \subseteq \hat{F}, F^\alpha \subseteq (\hat{F})^\alpha \). To show the reverse inclusion, let \( F \in (\hat{F})^\alpha \). Then there exists \( v \in \hat{F} \) and \( \beta < \alpha \) such that \( F = v^\beta \). Let \( 2\varepsilon = \alpha - \beta \) and \( \gamma = \beta + \varepsilon \). Then \( \mu = v + \varepsilon \in F \) and \( F = v^\beta = (v + \varepsilon)^{\beta + \varepsilon} = \mu^\gamma \subseteq F^\alpha \).

**Corollary 2.4.** Let \( F \) be a prefilter, \( c(F) = c > 0 \) and \( 0 < \alpha \leq c \). Then
\[
\hat{F} = \bigcap_{0 < \alpha \leq c} (\hat{F})^\alpha.
\]

**Proof.** \( \hat{F} \) is saturated and so, by ([15, Proposition 12]), we know that
\[
\hat{F} = \bigcap_{0 < \alpha \leq c} (\hat{F})^\alpha.
\]

Now, the result follows from the previous theorem.

3. Generalised filters

We call a nonzero function \( f : 2^X \rightarrow I \) a generalised filter (or a \( g \)-filter) on \( X \) iff

- **GF1** \( f(\emptyset) = 0; \)
- **GF2** \( \forall A, B \subseteq X, f(A \cap B) \geq f(A) \wedge f(B); \)
- **GF3** \( \forall A, B \subseteq X, A \subseteq B \Rightarrow f(A) \leq f(B); \)

Of course, the requirement that \( f \) be nonzero is equivalent to requiring that \( f(X) > 0 \).
For $f : 2^X \to I$ and $A \subseteq X$, we define

$$(f)(A) \overset{\text{def}}{=} \sup_{B \subseteq A} f(B).$$

If $f$ is nonzero and satisfies

GF1: $f(\emptyset) = 0$;

GF2: $\forall A, B \subseteq X$, $f(A) \wedge f(B) \leq (f)(A \cap B)$,

we shall call $f$ a generalised filter base (or a $g$-filter base) on $X$.

Naturally, a $g$-filter is a $g$-filter base. Furthermore,

**Theorem 3.1.** If $X$ is a set and $f$ is a $g$-filter base on $X$ then $(f)$ is a $g$-filter.

**Proof.** The conditions GF1 and GF3 are easily checked.

GF2: Let $A, B \subseteq X$. If \((f)(A) \wedge (f)(B) = 0\) then \((f)(A) \wedge (f)(B) \leq (f)(A \cap B)\). So let $\alpha < (f)(A) \wedge (f)(B)$.

Then

$$\begin{align*}
\alpha &< \sup_{U \subseteq A} f(U) \wedge \sup_{V \subseteq B} f(V) \\
&\Rightarrow \exists U \subseteq A, \exists V \subseteq B, \\
&\quad \alpha < f(U) \wedge f(V) \leq (f)(U \cap V) \\
&\Rightarrow \exists W \subseteq U \cap V \subseteq A \cap B, \quad \alpha < f(W) \\
&\Rightarrow \alpha < (f)(A \cap B).
\end{align*}$$

Thus, \((f)(A) \wedge (f)(B) \leq (f)(A \cap B)\). \hfill \Box

If $f$ is a $g$-filter base on $X$, we define the characteristic, $c(f)$, of $f$ by

$$c(f) = \sup_{A \subseteq X} f(A).$$

It follows from definition that $c(f) > 0$.

Just as for filters, we have:

**Lemma 3.2.** If $X$ is a set and $f$ is a $g$-filter base on $X$ then

$$c(f) = c((f)).$$

**Proof.**

$$c((f)) = \sup_{A \subseteq X} (f)(A) = \sup_{A \subseteq X} \sup_{B \subseteq A} f(B)$$

$$= \sup_{B \subseteq X} f(B) = c(f). \hfill \Box$$

The proof of the following lemma is straightforward and is left to the reader.

**Lemma 3.3.** Let $f$ be a $g$-filter on $X$ and let $A, B \subseteq X$.

Then

1. $c(f) = f(X)$;
2. $f(A \cap B) = f(A) \wedge f(B)$.

If $f$ is a $g$-filter (base) on $X$ with $c(f) = c$ then for $0 \leq \alpha < c$, we define the (upper) $\alpha$-level filter (base), $f^\alpha$, associated with $f$ by

$$f^\alpha \overset{\text{def}}{=} \{F \subseteq X : f(F) \geq \alpha\}$$

and for $0 < \alpha \leq c$, we define the (lower) $\alpha$-level filter (base), $f_\alpha$, associated with $f$ by

$$f_\alpha \overset{\text{def}}{=} \{F \subseteq X : f(F) \geq \alpha\}.$$

**Theorem 3.4.** If $f$ is a $g$-filter (base) on $X$ with $c(f) = c$ and:

(a) $0 \leq \alpha < c$ then $f^\alpha$ is a filter (base) on $X$;
(b) $0 < \alpha \leq c$ then $f_\alpha$ is a filter (base) on $X$.

**Proof.**

(a) Let $f$ be a $g$-filter on $X$. $f(X) = c > \alpha \Rightarrow X \in f^\alpha$.

Thus, $f^\alpha \neq \emptyset$.

If $F \in f^\alpha$ then $f(F) > \alpha \geq 0$, and hence, $F \neq \emptyset$.

If $A, B \in f^\alpha$ then $f(A) \wedge f(B) = f(A \cap B) > \alpha$, and hence, $A \cap B \in f^\alpha$.

Finally, if $A \in f^\alpha$ and $A \subseteq B$ then $f(B) \supseteq f(A) > \alpha$, and hence, $B \in f^\alpha$.

The proofs of the remaining three assertions are left to the reader. \hfill \Box

It is left as an easy exercise to show that the $\alpha$-level filters decrease as $\alpha$ increases and we record this as a lemma.

**Lemma 3.5.** If $f$ is a $g$-filter (base) with $c(f) = c$ and $0 \leq \alpha \leq \beta < c$ then

$$f_\alpha \subseteq f^\beta \subseteq f^\alpha \subseteq f^0.$$
Theorem 3.6. Let $X$ be a set, let $F(X)$ denote the collection of all filters on $X$ and let $G(X)$ denote the collection of all g-filters on $X$. Let

$$
\psi : F(X) \rightarrow G(X), \quad \mathbb{F} \mapsto 1_{\mathbb{F}}.
$$

Then $\psi$ is injective but not surjective.

Proof. The proof that $1_{\mathbb{F}}$ is a g-filter is left as an exercise.

To see that $\psi$ is not surjective it is sufficient to find a g-filter whose characteristic value is different from 1. $\square$

We note the following examples of g-filters, leaving the checking to the reader.

Examples 3.7.

(a) Let $X = \{1, 2, 3\}$ and define $f$ by

$$
\begin{align*}
  f(\mathbb{F}) &= 0 & \text{if } \mathbb{F} \notin \mathbb{F}, \\
  f(\{1\}) &= f(\{1, 3\}) = \frac{1}{4}, \\
  f(\{1, 2\}) &= \frac{3}{4}, \\
  f(\{1, 2, 3\}) &= 1.
\end{align*}
$$

(b) Let $U_n \overset{\text{def}}{=} \{m \in \mathbb{N} : m \geq n\}$ and define $g$ by

$$
\begin{align*}
g(F) &\overset{\text{def}}{=} \begin{cases} 
  \frac{1}{n} & \text{if } F = U_n, \\
  0 & \text{otherwise}
\end{cases}
\end{align*}
$$

and let $f = \langle g \rangle$.

(c) Let

$$
g_{x,a}(F) \overset{\text{def}}{=} \begin{cases} 
  a & \text{if } F = \{a\}, \\
  0 & \text{otherwise}
\end{cases}
$$

and let $f_{x,a} = \langle g_{x,a} \rangle$.

This g-filter has the special property that, for all $A, B \subseteq X$

$$f_{x,a}(A \cup B) = f_{x,a}(A) \lor f_{x,a}(B).$$

4. G-filters from prefilters

Let $\mathcal{F}$ be a prefilter on $X$ with $c(\mathcal{F}) = c > 0$. For $F \subseteq X$ define

$$S_{\mathcal{F}}(F) \overset{\text{def}}{=} \{x \in (0, c] : F \in \mathcal{F}^{x}\}.$$

Lemma 4.1. Let $\mathcal{F}$ be a prefilter with $c(\mathcal{F}) = c > 0$. Then $S_{\mathcal{F}}(F) = \emptyset$ or $S_{\mathcal{F}}(F)$ is an interval of form $(\beta, c]$.

Proof. If $S_{\mathcal{F}}(F) \neq \emptyset$ then there exists some $\alpha \in S_{\mathcal{F}}(F)$.

If $\alpha \leq \gamma \leq c$ then $F \in \mathcal{F}^{x} \subseteq \mathcal{F}^{\gamma}$ and so $\gamma \in S_{\mathcal{F}}(F)$.

Since

$$\mathcal{F}^{x} = \bigcup_{0 < \beta < x} \mathcal{F}^{\beta},$$

we have

$$\alpha \in S_{\mathcal{F}}(F) \Rightarrow F \in \mathcal{F}^{x} = \bigcup_{0 < \beta < x} \mathcal{F}^{\beta} \Rightarrow \exists \beta < \alpha, F \in \mathcal{F}^{\beta} \Rightarrow \exists \beta < \alpha, \beta \in S_{\mathcal{F}}(F).$$

This lemma allows us to define, for $F \subseteq X$

$$f_{\mathcal{F}}(F) \overset{\text{def}}{=} \begin{cases} 
  c - \inf_{\mathcal{F}} S_{\mathcal{F}}(F) & \text{if } S_{\mathcal{F}}(F) \neq \emptyset, \\
  0 & \text{if } S_{\mathcal{F}}(F) = \emptyset.
\end{cases}$$

We now need to check that the object defined above is indeed a g-filter.

Theorem 4.2. If $\mathcal{F}$ is a prefilter with $c(\mathcal{F}) = c > 0$ then $f_{\mathcal{F}}$ is a g-filter.

Proof.

(a)

$$\forall \alpha \leq c(\mathcal{F}), \emptyset \notin \mathcal{F}^{\alpha} \Rightarrow \forall \alpha \leq c(\mathcal{F}), \alpha \notin S_{\mathcal{F}}(\emptyset) \Rightarrow S_{\mathcal{F}}(\emptyset) = \emptyset \Rightarrow f_{\mathcal{F}}(\emptyset) = 0.$$

(b) Let $A, B \subseteq X$. Then

$$\alpha < f_{\mathcal{F}}(A) \land f_{\mathcal{F}}(B) \Rightarrow \inf_{\mathcal{F}} S_{\mathcal{F}}(A) < c - \alpha \text{ and } \inf_{\mathcal{F}} S_{\mathcal{F}}(B) < c - \alpha \Rightarrow A, B \in \mathcal{F}^{c - \alpha} \Rightarrow A \cap B \in \mathcal{F}^{c - \alpha} \Rightarrow c - \alpha \in S_{\mathcal{F}}(A \cap B) \Rightarrow \inf_{\mathcal{F}} S_{\mathcal{F}}(A \cap B) \leq c - \alpha \Rightarrow c - \inf_{\mathcal{F}} S_{\mathcal{F}}(A \cap B) = f_{\mathcal{F}}(A \cap B) \geq \alpha.$$
Thus,
\[ f_{\mathcal{F}}(A \cap B) \geq f_{\mathcal{F}}(A) \wedge f_{\mathcal{F}}(B). \]

(c) Let \( A \subseteq B \subseteq X \). Then
\[
\begin{align*}
f_{\mathcal{F}}(A) > \alpha & \Rightarrow c - \inf S_{\mathcal{F}}(A) > \alpha \\
& \Rightarrow \inf S_{\mathcal{F}}(A) < c - \alpha \\
& \Rightarrow c - \alpha \in S_{\mathcal{F}}(A) \\
& \Rightarrow A \in \mathcal{F}^{c-\alpha} \\
& \Rightarrow B \in \mathcal{F}^{c-\alpha} \\
& \Rightarrow c - \alpha \in S_{\mathcal{F}}(B) \\
& \Rightarrow \inf S_{\mathcal{F}}(B) \leq c - \alpha \\
& \Rightarrow c - \inf S_{\mathcal{F}}(B) = f_{\mathcal{F}}(B) \geq \alpha.
\end{align*}
\]

It follows that \( f_{\mathcal{F}}(A) \leq f_{\mathcal{F}}(B) \). □

If \( \mathcal{F} \) is a prefilter and \( \mu \in I^X \) we follow [18] and define the characteristic set of \( \mathcal{F} \) with respect to \( \mu \), denoted \( \mathcal{C}^\mu(\mathcal{F}) \), by
\[
\mathcal{C}^\mu(\mathcal{F}) = \{ \alpha \in I : \forall v \in \mathcal{F}, \exists x \in X, v(x) > \mu(x) + \alpha \} \\
= \{ \alpha \in I : \mu + \alpha \notin \mathcal{F} \}.
\]

The number \( c^\mu(\mathcal{F}) \de \sup \mathcal{C}^\mu(\mathcal{F}) \) will be called the characteristic value of \( \mathcal{F} \) with respect to \( \mu \). If \( F \subseteq X \) we write \( c^F(\mathcal{F}) \) instead of the more cumbersome \( c^{1\mathcal{F}}(\mathcal{F}) \).

**Theorem 4.3.** If \( \mathcal{F} \) is a prefilter with \( c(\mathcal{F}) = c > 0 \) and \( F \subseteq X \) then \( f_{\mathcal{F}}(F) = c - c^F(\mathcal{F}) \).

**Proof.** We just have to prove that \( c^F(\mathcal{F}) = \sup \mathcal{C}^F(\mathcal{F}) = \inf S_{\mathcal{F}}(F) \).

As usual, for \( F \subseteq X \), \( 1_F + \alpha \de (1_F + \alpha 1_X) \wedge 1 \) and it is easy to see that \( 1_F + \alpha = 1_X \vee 1_F \). Now,
\[
\begin{align*}
\inf S_{\mathcal{F}}(F) < \alpha \\
& \Rightarrow \alpha \in S_{\mathcal{F}}(F) \\
& \Rightarrow \exists \mu \in \mathcal{F}, \exists \beta < \alpha, \mu \beta = F \\
& \Rightarrow \exists \mu \in \mathcal{F}, \exists \beta < \alpha, \mu \beta = F \\
& \Rightarrow \exists \beta < \alpha, 1_F + \beta \in \mathcal{F} \\
& \Rightarrow \alpha \notin \mathcal{C}^F(\mathcal{F}) \\
& \Rightarrow c^F(\mathcal{F}) \leq \alpha.
\end{align*}
\]

Therefore, \( c^F(\mathcal{F}) \leq \inf S_{\mathcal{F}}(F) \). Furthermore,
\[
\begin{align*}
c^F(\mathcal{F}) < \alpha \\
& \Rightarrow \alpha \notin \mathcal{C}^F(\mathcal{F}) \\
& \Rightarrow 1_F + \alpha = \alpha 1_X \lor 1_F \in \mathcal{F} \\
& \Rightarrow \forall \beta > \alpha, (\alpha 1_X \lor 1_F)^\beta = F \in \mathcal{F}^c \beta \\
& \Rightarrow \forall \beta > \alpha, \beta \in S_{\mathcal{F}}(F) \\
& \Rightarrow \inf S_{\mathcal{F}}(F) \leq \alpha.
\end{align*}
\]

Thus, \( c^F(\mathcal{F}) = \inf S_{\mathcal{F}}(F) \). □

5. Prefilters from g-filters

Our next task is to show that a g-filter gives rise to a prefilter. However, we first discover the connection between the characteristic of a prefilter and the g-filter that it generates.

**Lemma 5.1.** If \( \mathcal{F} \) is a prefilter on \( X \) with \( c(\mathcal{F}) = c > 0 \) then:
\[
c(f_{\mathcal{F}}) = c(\mathcal{F}).
\]

**Proof.** Let \( c = c(\mathcal{F}) \). Then
\[
c(f_{\mathcal{F}}) = f_{\mathcal{F}}(X) = c - \inf S_{\mathcal{F}}(X).
\]

Now,
\[
\forall \alpha \leq c, X \in \mathcal{F} \Rightarrow \forall \beta \leq c, \alpha \in S_{\mathcal{F}}(X) \\
\Rightarrow \inf S_{\mathcal{F}}(X) = 0 \\
\Rightarrow c(f_{\mathcal{F}}) = f_{\mathcal{F}}(X) = c.
\]

For a g-filter \( f \) with \( c(f) = c \) we define
\[
\mathcal{F}_f \de \{ \nu \in I^X : \forall 0 < \alpha \leq c, \forall \beta < \alpha, \nu \beta \in f^{c-\alpha} \}.
\]

Of course, we need to check that this does produce a prefilter.

**Theorem 5.2.** If \( f \) is a g-filter then \( \mathcal{F}_f \) is a prefilter.

**Proof.**
(a) We observe that
\[
\forall \beta \leq c(f), 0^\beta = \{ x \in X : 0(x) > \beta \} = \emptyset.
\]
It follows that
\[ \forall 0 < \alpha \leq c, \forall \beta < \alpha, \ 0^\beta \in f^{c-\alpha} \]
and this means that \( 0 \notin \mathcal{F}_f \).

On the other hand, since \( c = c(f) = f(X) > 0 \) we have
\[ \forall 0 < \alpha \leq c, \forall \beta < \alpha, \ (1_X)^\beta = X \in f^{c-\alpha} \]
and so \( 1_X \in \mathcal{F}_f \).

(b) Let \( \mu, v \in \mathcal{F}_f \). It follows from
\[ \forall \beta < c(f), \ v^\beta \cap \mu^\beta = (v \cap \mu)^\beta \]
that \( v \land \mu \in \mathcal{F}_f \).

(c) Let \( v \in \mathcal{F} \) and \( v \leq \mu \). Let \( 0 < \alpha \leq c, \beta < \alpha \). Then \( v^\beta \in f^{c-\alpha} \). Since \( v^\beta \leq \mu^\beta \) we have
\[ f(\mu^\beta) \geq f(v^\beta) \geq c - \alpha. \]

Thus, \( \mu \in \mathcal{F}_f \). \( \Box \)

The following lemma simplifies some of the work later on.

Lemma 5.3. If \( f \) is a g-filter with \( c(f) = c > 0 \) then
\[ \mathcal{F}_f = \{ v \in I^X : \forall 0 \leq \gamma < c, \ v^\gamma \in f_{c-\gamma} \}. \]

Proof. Let us define
\[ \mathcal{G} \overset{\text{def}}{=} \{ v \in I^X : \forall 0 \leq \gamma < c, \ v^\gamma \in f_{c-\gamma} \}. \]

Let \( v \in \mathcal{F}_f \). To show that \( v \in \mathcal{G} \) let \( 0 \leq \gamma < c \). Choose \( \alpha \) such that \( \gamma < \alpha < c \). Then \( v^\gamma \in f^{c-\alpha} \). Since \( \alpha \) is arbitrary, we have
\[ \forall \alpha \in (\gamma, c), \ f(v^\alpha) > c - \alpha \]
and hence, \( f(v^\gamma) > c - \gamma \). In other words, \( v^\gamma \in f_{c-\gamma} \).

Conversely, let \( v \in \mathcal{G} \). To show that \( v \in \mathcal{F}_f \), let \( 0 < \alpha \leq c, \ 0 < \beta < \alpha \). Then we have \( 0 < \beta < c \). Thus, \( v^\beta \in f_{c-\beta} \) and so \( f(v^\beta) > c - \beta > c - \alpha \). Therefore, \( v^\beta \in f^{c-\alpha} \). \( \Box \)

The correlation between g-filters and prefilters is not completely straightforward. In fact, as we shall see, the prefilter associated with a g-filter is rather special.

Theorem 5.4. If \( f \) is a g-filter then the associated prefilter \( \mathcal{F}_f \) is saturated.

Proof. Suppose that
\[ \forall \varepsilon > 0, \ v + \varepsilon \in \mathcal{F}_f \]
We show that \( v \in \mathcal{F}_f \). To this end, we let \( \alpha < c(f) \) and \( \beta < \alpha \) and show that \( v^\beta \in f^{c-\alpha} \).

Choose \( \gamma \) such that \( \beta < \gamma < \alpha \) and let \( \varepsilon = \gamma - \beta \). Then, since \( v + \varepsilon \in \mathcal{F}_f \) we have
\[ (v + \varepsilon)^\gamma = (v + \varepsilon)^{\beta + \gamma} = v^\beta \in f^{c-\alpha}. \]

We saw, in Lemma 5.1, the connection between the characteristic of a prefilter and the g-filter that it generates. Let us now find the connection between a g-filter and the prefilter that it generates.

Theorem 5.5. Let \( f \) be a g-filter on \( X \). Then
\[ c(\mathcal{F}_f) = c(f). \]

Proof. Let \( c(f) = c \). Then
\[ \forall v \in \mathcal{F}_f, \forall 0 \leq \beta < \alpha \leq c, \ v^\beta \in f^{c-\alpha} \]
\[ \implies \forall v \in \mathcal{F}_f, \forall 0 \leq \beta < \alpha \leq c, \sup v > \beta \]
\[ \implies \forall v \in \mathcal{F}_f, \sup v \geq c \]
\[ \implies \inf v \leq c(f) \geq c. \]

On the other hand,
\[ \forall \alpha \leq c, \forall \beta < \alpha, \ (c_1)^\beta = X \in f^{c-\alpha} \]
and so \( c_1 \in \mathcal{F}_f \). Thus,
\[ c(\mathcal{F}_f) = \inf_\in \sup v \leq \sup \inf_\in = c. \]

The use of \( \alpha \)-level theorems has proved to be very useful in various situations. See, e.g. [26, 27, 17]. We therefore, investigate the \( \alpha \)-levels of g-filters.

Lemma 5.6. Let \( f \) be a g-filter with \( c(f) = c \) and let \( \alpha \in (0, c] \). Then
\[ \mathcal{F}_f^\alpha = f^{c-\alpha}. \]

Proof. Let \( F \in (\mathcal{F}_f)^\alpha \). Then there exists \( v \in \mathcal{F}_f \), \( \beta < \alpha \) such that \( F = v^\beta \). Since \( v \in \mathcal{F}_f \), we have \( v^\beta = F \in f^{c-\alpha} \).
Conversely, if \( F \in f^{c-}\) then \( f(F) \overset{\text{def}}{=} t > c - \alpha. \) Let \( v = (c - t)X + 1_F. \) We intend to invoke Lemma 5.3 to show that \( v \in \mathcal{T}. \) To this end, let \( 0 \leq \gamma < c. \)

If \( \gamma \in [c - t, c) \) then \( v^\gamma = F \) and so \( f(v^\gamma) = f(F) = t \geq c - \gamma. \)

If \( \gamma \in [0, c - t) \) then \( v^\gamma = X \) and so \( f(v^\gamma) = f(X) = c \geq c - \gamma. \)

We therefore have \( v^\gamma \in f_{c-\gamma} \) for all \( \gamma \in [0, c) \) and so \( v \in \mathcal{T} \) and \( F = v^{c-t} \) with \( c - t < \alpha. \) Thus, \( F \in \mathcal{T}^c. \)

\[ (f_{\mathcal{T}})^a = f_{\mathcal{T}^c-a}. \]

\[ A \in (f_{\mathcal{T}})^a \iff f_{\mathcal{T}}(A) = c - \inf S_\alpha(A) > \alpha \]
\[ \iff \inf S_\alpha(A) < c - \alpha \]
\[ \iff c - \alpha \in S_\alpha(A) \]
\[ \iff A \in \mathcal{T}^c-a. \]

We now establish the g-filter analogue of Theorem 2.1.4.

Lemma 5.8. If \( f \) is a \( \alpha \)-filter with \( c(f) = c \) and \( 0 \leq \alpha < c \) then

\[ f^\alpha = \bigcup_{\beta > \alpha} f^\beta. \]

\[ \text{Proof. If } F \in f^\alpha, \text{ choose } \beta \text{ such that } \alpha < \beta < f(F). \text{ Then } F \in f^\beta \text{ and so } F \in \bigcup_{\beta > \alpha} f^\beta. \]

Conversely, let \( F \in \bigcup_{\beta > \alpha} f^\beta. \) Then \( F \in f^\beta \) for some \( \beta > \alpha. \) Thus, \( f(F) > \beta > \alpha \) and hence, \( F \in f^\alpha. \)

Corollary 5.9. If \( f \) is a g-filter with \( c(f) = c \) and \( 0 < \alpha \leq c \) then

\[ f^{c-\beta} = \bigcup_{0 < \beta \leq \alpha} f^{c-\beta}. \]

In [2] we saw that saturated prefilters are specified by their \( \alpha \)-level filters. We show that a similar situation pertains for g-filters.

Lemma 5.10. If \( f \) and \( g \) are g-filters with \( c(f) \neq c(g) \) then \( f \neq g. \)

Lemma 5.11. Let \( f \) and \( g \) be g-filters with \( c(f) = c(g) = c. \) Then

\[ \forall \alpha < c, f^\alpha = g^\alpha \iff f = g. \]

\[ \text{Proof.} \]

\[ \forall \alpha < c, f^\alpha = g^\alpha \iff \forall \alpha < c, \forall A \subseteq X, A \in f^\alpha \iff A \in g^\alpha \]
\[ \iff \forall A \subseteq X, \forall \alpha \leq c, \]
\[ \quad (f(A) > \alpha \iff g(A) > \alpha) \]
\[ \iff \forall A \subseteq X, f(A) = g(A) \]
\[ \iff f = g. \]

We have seen that to each g-filter there corresponds a saturated prefilter and, conversely, to each prefilter there corresponds a g-filter. This inspires the following theorem:

Theorem 5.12. Let

\[ \mathcal{F}(X) \overset{\text{def}}{=} \{ \mathcal{F} \in 2^X : \mathcal{F} \text{ is a saturated prefilter on } X \} \]
\[ \mathcal{G}(X) \overset{\text{def}}{=} \{ f \in 2^X : f \text{ is a g-filter on } X \}. \]

Then \( \psi : \mathcal{F}(X) \to \mathcal{G}(X), \mathcal{F} \mapsto f_{\mathcal{F}} \) is a bijection.

\[ \text{Proof. We first show that } \psi \text{ is injective. To this end, let } \mathcal{F}, \mathcal{G} \in \mathcal{F}(X) \text{ with } \mathcal{F} \neq \mathcal{G}. \]

If \( c(\mathcal{F}) \neq c(\mathcal{G}) \) then

\[ f_{\mathcal{F}}(X) = c(\mathcal{F}) - \inf \{ \alpha \leq c(\mathcal{F}) : X \in \mathcal{F}^\alpha \} \]
\[ = c(\mathcal{F}) - 0 \]
\[ \neq c(\mathcal{G}) \]
\[ = f_{\mathcal{G}}(X) \]

and so \( f_{\mathcal{F}} \neq f_{\mathcal{G}}. \)

If \( c(\mathcal{F}) = c(\mathcal{G}) = c \) then

\[ \exists \alpha \leq c, \mathcal{F}^\alpha \neq \mathcal{G}^\alpha. \]

This follows from the fact that saturated prefilters are completely determined by their \( \alpha \)-level filters ([1, Theorem 2; 15, Theorem 11]). Suppose that
$F \in \mathcal{F}^x \setminus \mathcal{F}^y$. Then $a \in S_x(F) \setminus S_y(F)$. Thus, $\inf S_x(F) < a$ and $a \leq \inf S_y(F)$. Thus,

$$f_x(F) = c - \inf S_x(F) > c - a > c - \inf S_y(F) = f_y(F).$$

So, once again, $f_x \neq f_y$.

In order to show that $\psi$ is surjective, let $f \in \mathcal{F}(X)$ and let $c(\mathcal{F}) = c$. Then

$$\mathcal{F}_f = \{ v \in I^X; \forall \alpha \leq c, \forall \beta < \alpha, \forall \theta \in f^{c-\alpha} \}.$$

Now, appealing to Lemmas 5.6 and 5.7, we have

$$\forall \alpha \in [0, c), (f(\mathcal{F})^2 = (f(\mathcal{F}))^{c-\alpha} = f^{c-\alpha} = f^x.$$

It therefore follows from Lemma 5.11 that

$$\psi(\mathcal{F}_f) = f(\mathcal{F}) = f.$$  \quad \Box

We extract the following corollaries:

**Corollary 5.13.** If $f$ is a $g$-filter on $X$ then

$$f(\mathcal{F}) = f.$$  

**Corollary 5.14.** If $\mathcal{F}$ is a saturated prefilter on $X$ then

$$\mathcal{F}_{f(\mathcal{F})} = \mathcal{F}.$$  

**Proof.** Let $\psi: \mathcal{F} \to \mathcal{G}$ as in the theorem. Then

$$\psi(\mathcal{F}) = f(\mathcal{F})$$  

and

$$\psi(\mathcal{F}_{f(\mathcal{F})}) = f(\mathcal{F}) = f(\mathcal{F}).$$

Thus, it follows from the injectivity of $\psi$ that

$$\mathcal{F} = \mathcal{F}_{f(\mathcal{F})}. \quad \Box$$

We have developed the $g$-filter analogues of various prefilter notions and it is natural, therefore, to seek a $g$-filter analogue of the saturation operator. In other words, if $f$ is a $g$-filter, we seek a definition of $\hat{f}$, the saturation of $f$, which is consistent with the theory which we have developed thus far. We would require, among other things, that the saturation of the $g$-filter associated with a prefilter is the $g$-filter associated with the saturation of the prefilter. In symbols

$$\hat{f} = f(\mathcal{F}).$$

However, we have the following lemma:

**Lemma 5.15.** If $\mathcal{F}$ is a prefilter then

$$f(\mathcal{F}) = f_y(\mathcal{F}).$$

**Proof.** For $\mathcal{F} \subseteq \mathcal{F}$:

$$f_y(\mathcal{F}) = c(\mathcal{F}) - \inf \{ v; F \in (\mathcal{F})^y \} = c(\mathcal{F}) - \inf \{ v; F \in (\mathcal{F})^y \} = f_y(\mathcal{F}). \quad \Box$$

Thus, for a prefilter $\mathcal{F}$

$$\hat{f} = f(\mathcal{F}).$$

The most natural definition of $\hat{f}$ which accomplishes this is the simple

$$\hat{f} \overset{\text{def}}{=} f.$$  

In this sense, $g$-filters are already saturated. This explains why, in [7], the definition of a generalised uniformity did not include a saturation condition analogous to FU2 in Definition 2.1 of [20]. The situation is also illustrated by the following theorem which extends Theorem 5.4.

**Theorem 5.16.** If $\mathcal{F}$ is a prefilter then

$$\mathcal{F}_{f(\mathcal{F})} = \mathcal{F}.$$  

**Proof.** From Theorem 5.4 we know that $\mathcal{F}_{f(\mathcal{F})}$ is a saturated prefilter and so, according to [15, Theorem 11], we must show that

$$\forall \alpha \leq c(\mathcal{F}) = c(\mathcal{F}), (\mathcal{F}_{f(\mathcal{F})})^x = (\mathcal{F})^x.$$  

Now,

$$(\mathcal{F}_{f(\mathcal{F})})^x = (f(\mathcal{F}))^{c-\alpha} = \mathcal{F}^x = \mathcal{F}.$$  \quad \Box$$

From this last result we obtain the following characterisation of the saturation of a prefilter.

**Corollary 5.17.** If $\mathcal{F}$ is a prefilter with $x(\mathcal{F}) = x > 0$, then

$$\mathcal{F} = \{ v \in I^X; \forall 0 < \alpha \leq c, \forall \beta < \alpha, \forall \theta \in (\mathcal{F})^x \}. $$
**Proof.**

\[ v \in \mathcal{F} \iff v \in \mathcal{F}_f \]

\[ \iff \forall 0 < \alpha \leq c, \forall \beta < \alpha, \]

\[ v^\beta \in (f^-)^{c-\alpha} = \mathcal{F}^c. \]

\[ \square \]

This concludes our study of the basic properties of generalised filters. In [8], we investigate prime g-filters and the behaviour of images and preimages of g-filters.

**Acknowledgements**

We would like to thank the Rhodes University Research Council and the Foundation for Research Development for their generous financial support. We also gratefully acknowledge the helpful comments of Prof. M.A. de Prada Vicente.

**References**


