"As I made my way home,
I thought Jem and I would get grown
but there wasn't much else left for
us to learn, except possibly algebra."

Harper Lee, To Kill A Mockingbird

The Automorphism Group of a Free Group is not Subgroup Separable

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To Joan Birman
with a deep feeling of admiration

Abstract

By a classical result of Gilbert Baumslag, the automorphism group $\mathrm{Aut}(G)$ of a finitely generated residually finite group G is residually finite. While this implies that the automorphism group of a free group F_n of finite rank n is residually finite, the aim of this paper is to show that, for a (non-abelian) free group, this group is not subgroup separable. As a corollary of our proof, we will get that the braid groups B_n, $n \geq 4$, are not subgroup separable. In contrast, the groups B_2 and B_3 are well-known to have this property.

The automorphism group $\mathrm{Aut}(F_2)$ is still not well understood. For example, while it is known [FP92] that $\mathrm{Aut}(F_n)$ is not linear if $n > 2$, this question is open for $\mathrm{Aut}(F_2)$ and is equivalent to a problem for the braid groups: The braid group B_4 is linear (over $\mathbb{C}$) if and only if $\mathrm{Aut}(F_2)$ is linear [DFG82]. In an additional section, we will clarify this result to show it is true over any field, rather than just over $\mathbb{C}$. The braid groups B_3 and B_2 are known to be subgroups of $\mathrm{SL}_2(\mathbb{C})$.

Our main goal is to prove that $\mathrm{Aut}(F_2)$ and thus $\mathrm{Aut}(F_n)$, $n \geq 3$, is not subgroup separable. Our hope is that this can help to exclude certain representations of $\mathrm{Aut}(F_2)$ or B_4 of being faithful.

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Definition 1.1. *A group G is subgroup separable or LERF (locally extended residually finite) if, for each finitely generated subgroup $H \subset G$ and each element $g \in G$ which does not lie in H, there is a subgroup of finite index in G that contains H but does not contain g. In particular, a subgroup separable group is residually finite.*

The property of subgroup separability is of special interest in combinatorial group theory because of its implications for decision problems in finitely presented groups. As for a finitely presented group, G, residually finiteness means that the word problem is solvable, it is easy to see that subgroup separability means that the generalized word problem for G is solvable. That is, it can be decided for every finitely generated subgroup H of G and every element $g \in G$ whether or not g is in H.

Subgroup separability is also of particular importance in geometric topology. A theorem of Scott ([Sco78]) equates subgroup separability in a group G with a purely topological property if G is the covering group for a regular cover, $p : \tilde{X} \to X$, of a Hausdorff topological space. The group G is subgroup separable if and only if for any finitely generated $H < G$ and any compact subset $C \subset \tilde{X}/H$, there is a finite cover X_f of X such that the projection from $\tilde{X}/H$ to X factors through X_f so that C maps by a homeomorphism into X_f. In particular, this implies that surfaces incompressibly immersed in a three-manifold with subgroup separable fundamental group lift to embedded surfaces in a finite cover of the three-manifold. Therefore, subgroup separability has direct relevance to the virtually Haken conjecture and the positive virtual Betti number conjecture for hyperbolic three-manifolds (see e.g. [Lon88]).

In general, the property of subgroup separability seems to be difficult to prove, and, as yet, there is not a single example of a closed hyperbolic three-manifold for which it is known whether or not the fundamental group is subgroup separable. Among groups known to be subgroup separable are the free groups ([HJ49]), surface groups and the fundamental groups of Seifert fibered three-manifolds ([Sco78]). Moreover, it follows easily from the definition that every subgroup and every finite extension of a subgroup separable group is subgroup separable (see e.g. [Sco78]).

As a further positive result one has:

Proposition 1.2. *The group $F_m \times \mathbb{Z}$ is subgroup separable (see e.g. [AG73]). The braid group B_3 on three strands contains as a subgroup of index 6 the pure braid group $P_3 \cong F_2 \times \mathbb{Z}$, and thus B_3 is also subgroup separable.*

Since B_2 is isomorphic to $\mathbb{Z}$, it is obviously subgroup separable.

As a negative result, it is known that, in general, the direct product of two subgroup separable groups need not be subgroup separable. By a result of Mihailova (see [MI71] and compare with [Gru78]), the group $F_2 \times F_2$ is the most prominent counterexample. Furthermore, in [BKS87], it is shown that the fundamental group of a certain 3-manifold is not subgroup separable.

Proposition 1.3 ([BKS87]). *The group*

$$K = \langle y, \alpha, \beta \,|\, y^{-1}\alpha y = \alpha\beta, y^{-1}\beta y = \beta \rangle$$

is not subgroup separable.

We will use this example to show that the automorphism group $\mathrm{Aut}(F_2)$ of a free group of rank 2 is not subgroup separable. Therefore, the automorphism group of any non-abelian free group cannot be subgroup separable. Remember that, by definition (or by proof as the reader might see the braid groups), the groups B_n are subgroups of $\mathrm{Aut}(F_n)$ (see e.g. [Bir74]).

Theorem 1.4. *The automorphism group of a nonabelian free group is not subgroup separable.*

Proof. We will show the result for $\mathrm{Aut}(F_2)$. Since $\mathrm{Aut}(F_2)$ is contained in any $\mathrm{Aut}(F_n)$ for n greater than 2 the result follows. In [DFG82] it is shown that the braid group B_4 modulo its center C_4 - which is well known to be infinite cyclic - is isomorphic to a subgroup of index 2 in $\mathrm{Aut}(F_2)$. Hence we are done if we can prove that B_4/C_4 is not subgroup separable.

It is shown in [MS97] that the fundamental group of any orientable punctured torus bundle over the circle is isomorphic to a subgroup of B_4/C_4 as long as no nontrivial power of the monodromy map induces an inner automorphism on the fundamental group of the fiber. The group K of Proposition 1.3 is the fundamental group of the punctured torus bundle with monodromy map given by a single Dehn twist about a meridional curve. Therefore K is a subgroup of B_4/C_4. Specifically, the homomorphism $h : K \to B_4/C_4$ given by

$$\begin{aligned} h(y) &= \sigma_2\sigma_3^{-1}\sigma_2^{-1}C_4 \\ h(\alpha) &= \sigma_1\sigma_3^{-1}C_4 \\ h(\beta) &= \sigma_2\sigma_1\sigma_3^{-1}\sigma_2^{-1}C_4 \end{aligned}$$

is injective. Here we use Artin's presentation of the braid group (see e.g. [Bir74]). Since K is not subgroup separable, we are done. □

Before we can go on, we need a lemma which we were not able to find in the published literature. It, however, already appeared in the unpublished notes of A. Newworld [New]. Since these notes are not widely known, we will add a proof of Newworld's lemma.

Lemma 1.5 (A. Newworld). *The pure braid group P_n is the direct product of its center $C_n \cong \mathbb{Z}$ (which is also the center of B_n) and a subgroup isomorphic to P_n/C_n. (Note that this statement is not true for the braid group itself. The quotient B_n/C_n is not a subgroup of B_n for $n \geq 3$.)*

Proof. The pure braid group P_n is generated by the elements $a_{i,j} := \sigma_i\sigma_{i+1}\cdots\sigma_{j-1}^2\sigma_{j-2}^{-1}\cdots\sigma_i^{-1}$, $1 \leq i < j \leq n$, and its center C_n is infinite cyclic generated by

$$(a_{1,n}a_{1,n-1}\cdots a_{1,2})\cdots(a_{n-2,n}a_{n-2,n-1})(a_{n-1,n}).$$

In particular, since $a_{n-1,n}$ appears exactly once in the generator of C_n, we have that P_n is the product of its subgroup $H := \langle a_{i,j},\ (i,j) \neq (n-1,n)\rangle$ and C_n.

Since C_n commutes with H, it remains to show that H and C_n are disjoint. This can be seen by the following argument: the commutator factor group $H_1(P_n,\mathbb{Z})$ is free abelian of rank $n(n-1)/2$ and is generated by the images of the $n(n-1)/2$ generators $a_{i,j}$. Hence C_n maps isomorphically to $H_1(P_n,\mathbb{Z})$ of P_n, and the image of H in $H_1(P_n,\mathbb{Z})$ must be disjoint from the image of C_n. Therefore, H and C_n must be disjoint, and we get that $P_n \cong H \times C_n$ and $H \cong P_n/C_n$. □

By Newworld's lemma, P_4 is a direct product $P_4/C_4 \times C_4$. Thus P_4/C_4, which is not subgroup separable since it is a finite index subgroup of B_4/C_4 and, hence, of $\mathrm{Aut}(F_2)$, is a subgroup of B_4. We get, with the help of Proposition 1.2, as a corollary:

Corollary 1.6. *The groups B_n are only subgroup separable for $n = 2$ or $n = 3$.*

Remark 1.7. *As Sergei Chmutov pointed out to us, it was already shown in [Mak81] that the generalized word problem (also known as occurrence problem) is not solvable for B_n, $n \geq 5$, and, thus, these groups cannot be subgroup separable. The proof uses the result of Mihailova*

([MI71]) that the direct product $F_2 \times F_2$ does not admit a solution to the generalized word problem. It is quite easy to see that $F_2 \times F_2$ is contained in B_5. In contrast to this result, in [Aki91] it is shown that B_4 does not contain $F_2 \times F_2$.

It remains an open problem whether B_4 admits a solution to the generalized word problem or not.

2 The groups B_4 and Aut(F_2)

The aim of this section is to clarify the proof of Dyer, Formanek and Grossman that B_4 is linear over $\mathbb{C}$ if and only if $\mathrm{Aut}(F_2)$ is. We will give a simple argument that the restriction on the field is not needed.

Theorem 2.1. *The group B_4 is linear (over any field) if and only if $Aut(F_2)$ is.*

Proof. By elementary representation theory we know that if a subgroup of finite index of a group is linear, then the group itself is linear. In [DFG82] it is shown that B_4 modulo its center $C_4 \cong \mathbb{Z}$ is a subgroup of index 2 in $\mathrm{Aut}(F_2)$. Now C_4 is a subgroup of the pure braid group P_4 of index 24 in B_4, and, by Newworld's lemma, P_4 is the direct product $P_4/C_4 \times C_4$. Since $\mathbb{Z}$ is also contained in P_4/C_4, we get that P_4 and, thus, B_4 is linear if and only if P_4/C_4 and, thus, $\mathrm{Aut}(F_2)$ is. □

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