

# Ideals with Large( $r$ ) Projective Dimension and Stillman's Question

Jason McCullough



October 22, 2011  
AMS Sectional Meeting  
Special Session on Commutative Algebra  
University of Utah

# Notation

$K$  a field

$$S = K[X_1, X_2, \dots, X_n]$$

$$S = \bigoplus_{i=0}^{\infty} S_i \text{ is graded}$$

$S(-d)_i = S_{i-d} =$  rank one free module with generator in degree  $d$

$I = (f_1, \dots, f_t) \subset S$  a homogeneous ideal

(i.e. each  $f_j$  is in some  $S_i$ )

# Graded Free Resolutions

Compute a graded free resolution of  $S/I$ :

$$0 \leftarrow S/I \leftarrow S \leftarrow \bigoplus_j S(-d_j)^{\beta_{1j}} \leftarrow \bigoplus_j S(-d_j)^{\beta_{2j}} \leftarrow \cdots \leftarrow \bigoplus_j S(-d_j)^{\beta_{pj}} \leftarrow 0$$

# Graded Free Resolutions

Compute a graded free resolution of  $S/I$ :

$$0 \leftarrow S/I \leftarrow S \leftarrow \bigoplus_j S(-d_j)^{\beta_{1j}} \leftarrow \bigoplus_j S(-d_j)^{\beta_{2j}} \leftarrow \cdots \leftarrow \bigoplus_j S(-d_j)^{\beta_{pj}} \leftarrow 0$$

$$\text{Projective Dimension of } S/I = \text{pd}(S/I) = \max\{i \mid \beta_{ij} \neq 0\}$$

# Graded Free Resolutions

Compute a graded free resolution of  $S/I$ :

$$0 \leftarrow S/I \leftarrow S \leftarrow \bigoplus_j S(-d_j)^{\beta_{1j}} \leftarrow \bigoplus_j S(-d_j)^{\beta_{2j}} \leftarrow \cdots \leftarrow \bigoplus_j S(-d_j)^{\beta_{pj}} \leftarrow 0$$

Projective Dimension of  $S/I = \text{pd}(S/I) = \max\{i \mid \beta_{ij} \neq 0\}$

Regularity of  $S/I = \text{reg}(S/I) = \max\{j - i \mid \beta_{ij} \neq 0\}$

# Betti Diagrams

Record the Betti numbers  $\beta_{ij}$  in a matrix called the Betti table:

	0	1	2	...	i	...
0:	$\beta_{0,0}$	$\beta_{1,1}$	$\beta_{2,2}$	...	$\beta_{i,i}$	...
1:	$\beta_{0,1}$	$\beta_{1,2}$	$\beta_{2,3}$	...	$\beta_{i,i+1}$	...
2:	$\beta_{0,2}$	$\beta_{1,3}$	$\beta_{2,4}$	...	$\beta_{i,i+2}$	...
⋮	⋮	⋮	⋮		⋮	
j:	$\beta_{0,j}$	$\beta_{1,j+1}$	$\beta_{2,j+2}$	...	$\beta_{i,i+j}$	...
⋮	⋮	⋮	⋮		⋮	

$\text{pd}(S/I)$  = last nonzero column in Betti table.

$\text{reg}(S/I)$  = last nonzero row in Betti table.

# Burch-Kohn Theorem (Polynomial Ring Case)

## Theorem

*For any  $n \in \mathbb{N}$ , there is a three-generated ideal  $I = (f, g, h)$  in a polynomial ring  $S = K[x_1, \dots, x_{2n}]$  with  $\text{pd}(S/I) = n$ .*

## Remark

Engheta computed the degrees of the 3 generators:  
 $n - 2, n - 2, 2n - 2$ . (linear growth with respect to  $n$ )

# Stillman's Question

## Question (Stillman)

*Is there a bound, independent of  $n$ , on the projective dimension of ideals in  $S = K[X_1, \dots, X_n]$  which are generated by  $N$  homogeneous polynomials of given degrees  $d_1, \dots, d_N$ ?*

## Remark

Hilbert Syzygy Theorem guarantees  $\text{pd}(S/I) \leq n$ , but we seek a bound independent of  $n$ .

# Stillman's Question

## Known Cases:

- 1  $I = (m_1, \dots, m_N)$  is a monomial ideal. Then  $\text{pd}(S/I) \leq N$ .  
(Taylor resolution)
- 2  $I = (f, g, h)$  with  $d_1 = d_2 = d_3 = 2$ . Then  $\text{pd}(S/I) \leq 4$ .  
(Eisenbud-Huneke) e.g.:  $I = (w^2, x^2, wy + xz)$  satisfies  $\text{pd}(S/I) = 4$ . So bound is tight.
- 3  $I = (f, g, h)$  with  $d_1 = d_2 = d_3 = 3$ . Then  $\text{pd}(S/I) \leq 36$ .  
(Engheta)  
Largest known example:  $I = (x^3, y^3, x^2a + xyb + y^2c)$  satisfies  $\text{pd}(S/I) = 5$ . Bound is likely not tight.
- 4  $I = (f_1, \dots, f_N)$  with  $d_1 = \dots = d_N = 2$ . Then  $\text{pd}(S/I) \leq 2N^{2N}$ . (Ananyan-Hochster)

# Zhang's Question

## Question (Zhang)

Given a homogeneous ideal  $I$  in  $S = K[X_1, \dots, X_n]$  with  $N$  generators of degrees  $d_1, \dots, d_N$ , is

$$\text{pd}(S/I) \leq \sum_{i=1}^N d_i?$$

The following construction shows that this bound is too small.

# Definition

Fix integers  $m, n, d$  such that  $m \geq 1$ ,  $n \geq 0$  and  $d \geq 2$ .

Order the  $M_{m,d-1} = \frac{(m+d-2)!}{(m-1)!(d-1)!}$  monomials of degree  $d-1$  over the variables  $X_1, \dots, X_m$ .

$Z_i = i^{\text{th}}$  such monomial

$p = M_{m,d-1}$

$S = K[X_1, \dots, X_m, Y_{1,1}, \dots, Y_{p,n}]$

$$I_{m,n,d} = \left( X_1^d, \dots, X_m^d, \sum_{j=1}^p Z_j Y_{j,1}, \dots, \sum_{j=1}^p Z_j Y_{j,n} \right)$$

$m + n$  degree- $d$  generators

## Theorem (-)

$$\text{pd}(R/I_{m,n,d}) = m + np = m + n \frac{(m+d-2)!}{(m-1)!(d-1)!}.$$

# Proof Sketch

## Theorem (Homogeneous Auslander-Buchsbaum)

For any graded  $S$ -module  $M$ ,

$$\text{depth } M + \text{pd } M = \text{depth } S$$

where  $\text{depth } M = \text{length of maximal regular sequence on } M \text{ in the graded maximal ideal.}$

$$\text{pd}(S/I_{m,n,d}) = \dim S \Leftrightarrow \text{depth}(S/I_{m,n,d}) = 0$$

$\Leftrightarrow \exists s \in S - I$  with  $Xs \in I$  for every variable  $X$  of  $S$ .

Check that  $X_1^{d-1} X_2^{d-1} \cdots X_m^{d-1}$  works.

Example:  $I = I_{3,4,2}$

$$S = K[X_1, \dots, X_3, Y_{1,1}, \dots, Y_{3,4}]$$

$$I = (X_1^2, X_2^2, X_3^2, X_1 Y_{1,1} + X_2 Y_{2,1} + X_3 Y_{3,1}, X_1 Y_{1,2} + X_2 Y_{2,2} + X_3 Y_{3,2}, X_1 Y_{1,3} + X_2 Y_{2,3} + X_3 Y_{3,3}, X_1 Y_{1,4} + X_2 Y_{2,4} + X_3 Y_{3,4})$$

7 quadratic generators

$$\text{pd}(S/I) = \# \text{variables} = 15$$

# Three-Generated Case: $I = I_{2,1,d}$

$$S = K[X_1, X_2, Y_1, \dots, Y_d]$$

$$I = (X_1^d, X_2^d, X_1^{d-1}Y_1 + X_1^{d-2}X_2Y_2 + X_1^{d-3}X_2^2Y_3 + \dots + X_2^{d-1}Y_d)$$

Three generators in degree  $d$ .

$$\text{pd}(S/I) = d + 2$$

## Question

*Can we find a three-generated ideal with  $\text{pd}(S/I) > 3d$ ?*

# Definition

Fix integers  $g \geq 2$ ,  $m_1, \dots, m_n$  with  $m_n \geq 0$ ,  $m_{n-1} \geq 1$ , and  $m_i \geq 2$  for  $i < n$ .

We define an ideal with  $g + 1$  generators of degree  $d = 1 + \sum_i m_i$ .

Notation:

$$\mathcal{A}_k = \left\{ (a_{j,k'}) \left| \begin{array}{l} \text{Only first } k \text{ columns are nonzero} \\ \sum_{j=1}^g a_{j,i} = m_i \text{ with at least 2 nonzero en-} \\ \text{tries for } i \leq k \\ \text{(or just } \sum_{j=1}^g a_{j,n} = m_i \text{ when } i = n \leq k.) \end{array} \right. \right\}$$

# Definition

$$S = K[X, y_A \mid X = (x_{j,k}), A \in \mathcal{A}_n],$$

$$I_{g, (m_1, \dots, m_n)} = (x_{1,1}^d, \dots, x_{g,1}^d, f),$$

where

$$f = \sum_{k=1}^{n-1} \sum_{A \in \mathcal{A}_{k-1}} \sum_{j=1}^g X^A x_{j,k}^{m_k} x_{j,k+1}^{d_{k+1}} + \sum_{B \in \mathcal{A}_n} X^B y_B.$$

$$(d_k = m_k + \dots + m_n + 1)$$

## Theorem (Beder, Nunez, Seceleanu, Snapp, Stone, -)

*Using the notation above with  $I = I_{g, (m_1, \dots, m_n)}$ ,  $\text{depth}(R/I) = 0$ .*

Proof Sketch: Set  $T = (t_{j,k})$  and  $t_{j,k} = d_k - 1$ .

Set  $S = X^T$ .

Show  $S \in (I : \mathfrak{m}) - I$ .

## Corollary

$$\text{pd}(R/I) = \prod_{i=1}^{n-1} \left( \frac{(m_i + g - 1)!}{(g - 1)!(m_i)!} - g \right) \left( \frac{(m_n + g - 1)!}{(g - 1)!(m_n)!} \right) + gn.$$

Proof: Count the variables.  $g \times n$   $X$  variables and  $|\mathcal{A}_n|$   $Y$  variables.

Example:  $I = I_{2,(3,1)}$

$$S = K \left[ x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}, y \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, y \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, y \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}, y \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \right]$$

$$I = (x_{1,1}^5, x_{2,1}^5, x_{1,1}^3 x_{1,2}^2 + x_{2,1}^3 x_{2,2}^2 + x_{1,1}^2 x_{1,2} x_{2,1} y \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} + x_{1,1} x_{2,1}^2 x_{2,2} y \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \\ + x_{1,1} x_{1,2} x_{2,1}^2 y \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} + x_{1,1}^2 x_{2,1} x_{2,2} y \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix})$$

By previous Theorem,

$$\text{pd}(S/I) = \left( \frac{(3+2-1)!}{(2-1)!(3)!} - 2 \right) \left( \frac{(1+2-1)!}{(2-1)!(1)!} \right) + 2 \cdot 2 = 8.$$

Example:  $I = I_{2,(3,1)}$

	0	1	2	3	4	5	6	7	8
total:	1	3	53	184	287	248	124	34	4
0:	1	-	-	-	-	-	-	-	-
1:	-	-	-	-	-	-	-	-	-
2:	-	-	-	-	-	-	-	-	-
3:	-	-	-	-	-	-	-	-	-
4:	-	3	-	-	-	-	-	-	-
5:	-	-	-	-	-	-	-	-	-
6:	-	-	-	-	-	-	-	-	-
7:	-	-	-	-	-	-	-	-	-
8:	-	-	3	-	-	-	-	-	-
9:	-	-	3	4	-	-	-	-	-
10:	-	-	13	46	68	56	28	8	1
11:	-	-	33	132	218	192	96	26	3
12:	-	-	1	2	1	-	-	-	-

## Corollary

Over any field  $K$  and for any positive integer  $p$ , there exists an ideal  $I$  in a polynomial ring  $S$  over  $K$  with three homogeneous generators in degree  $p^2$  such that  $\text{pd}(R/I) \geq p^{p-1}$ .

Proof:

$$I = I_{2, \underbrace{(p+1, \dots, p+1, 0)}_{p-1 \text{ times}}}$$

## Corollary

*Over any field  $K$  and for any positive integer  $p$ , there exists an ideal  $I$  in a polynomial ring  $R$  over  $K$  with  $2p + 1$  homogeneous generators in degree  $2p + 1$  such that  $\text{pd}(R/I) \geq p^{2p}$ .*

Proof:

$$I_{2p, \underbrace{(2, 2, 2, \dots, 2)}_{p \text{ times}}}.$$

Note: The first family  $I_{m,1,d}$  is a subfamily of the new family.

$$I_{m,1,d} = I_{m,(d-1)}$$

(up to a relabeling of the variables)

Let

$$C_d = (w^d, x^d, wy^{d-1} + xz^{d-1}) \subset S = K[w, x, y, z]$$

Caviglia showed  $\text{reg}(S/C_d) = d^2 - 1$ .

$C_d$  is also a subfamily the new family.

$$C_d = I_{2,(1,d-2)}$$

(up to a relabeling of the variables)

## Question

What is  $\text{reg}(S/I_{g,(m_1,\dots,m_n)})$ ?

Caviglia's family has quadratic regularity growth relative to degree of the 3 gens. Some of our ideals have larger regularity...

Example:  $I = I_{2,(2,1,2)}$

$$S = K [x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}, x_{2,2}, x_{2,3}]$$

$$I = (x_{1,1}^6, x_{2,1}^6, x_{1,1}^2 x_{1,2}^4 + x_{2,1}^2 x_{2,2}^4 + x_{1,1} x_{2,1} x_{1,2} x_{1,3}^3 + x_{1,1} x_{2,1} x_{2,2} x_{2,3}^3)$$

$I$  has three degree 6 generators and  $\text{reg}(S/I) = 41$ .

# Stillman's Question - Regularity Version

## Question (Stillman)

*Is there a bound, independent of  $n$ , on the **regularity** of ideals in  $S = K[X_1, \dots, X_n]$  which are generated by  $N$  homogeneous polynomials of given degrees  $d_1, \dots, d_N$ ?*

Caviglia proved: This question  $\Leftrightarrow$  PD question.  
(See Bahman Engheta's thesis.)

# Questions

Note: Socle elements grow linearly with degree.

$\Rightarrow \operatorname{reg}(S/I_{2,(d-1)}) \geq 2d - 2$ . Caviglia showed

$\operatorname{reg}(S/I_{2,(1,d-2)}) \geq d^2 - 1$ .

## Question

*Does  $\operatorname{reg}(S/I_{2,(2,1,d)})$  exhibit cubic growth?*

## Question

*Does  $\operatorname{reg}(S/I_{2,(2,2,2,2,\dots,2)})$  exhibit exponential growth?*

## Thank you!

### References:

- T. Ananyan, M. Hochster, "Ideals Generated by Quadratic Polynomials." arXiv:1106.0839v1 [math.AC].
- J. Caviglia, "Koszul Algebras, Castelnuovo-Mumford Regularity, and Generic Initial Ideals." PhD Thesis. U. of Kansas. 2005
- B. Engheta, "Bounding Projective Dimension." PhD Thesis. U. of Kansas. 2005
- J. McCullough, "A Family of Ideals with Few Generators in Low Degree and Large Projective Dimension." Proceedings of the AMS. Volume 139 (2011), No. 6. pp. 2017-2023.
- J Beder, J McCullough, L Nunez, A Seceleanu, B Snapp, B Stone, "Ideals with Larger Projective Dimension and Regularity." Journal of Symbolic Computation 46 (2011) pp. 1105-1113.