Lifting Splittings and the Strong Direct Summand Conjecture

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For this talk, all rings are:

- commutative
- local (Noetherian)
- unital

Vanishing Conjecture for Maps of Tor (VCMT)

Conjecture (Hochster-Huneke)

Let $R \to A \to S$ be maps of rings such that R and S are regular and such that A is module-finite over R. Let M be any R-module. Then the maps

$$\operatorname{Tor}_i^R(M, A) \to \operatorname{Tor}_i^R(M, S)$$

are zero for all $i \geq 1$.

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  2. $M = R/I$, $S = R/xR$ and $x \in I$, or x is a nonzerodivisor on R/I. (Ranganathan)
  3. Open in mixed characteristic.

Strong Direct Summand Conjecture (SDSC)

Conjecture (Ranganathan)

Let $(R, \mathfrak{m})$ be a regular local ring and let A be a module finite extension of R. Let $x \in \mathfrak{m} - \mathfrak{m}^2$ and let $Q \subset A$ be a height one prime lying over xR. Then $xR \to Q$ splits as a map of R-modules. (i.e. $Q \simeq xR \oplus T$ some R-module T.)

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Closely related to the Direct Summand Conjecture (DSC):

Conjecture (Hochster)

In the setting above, $R \to A$ splits as a map of R-modules. (i.e. $A = R \oplus U$ some R-module U.)

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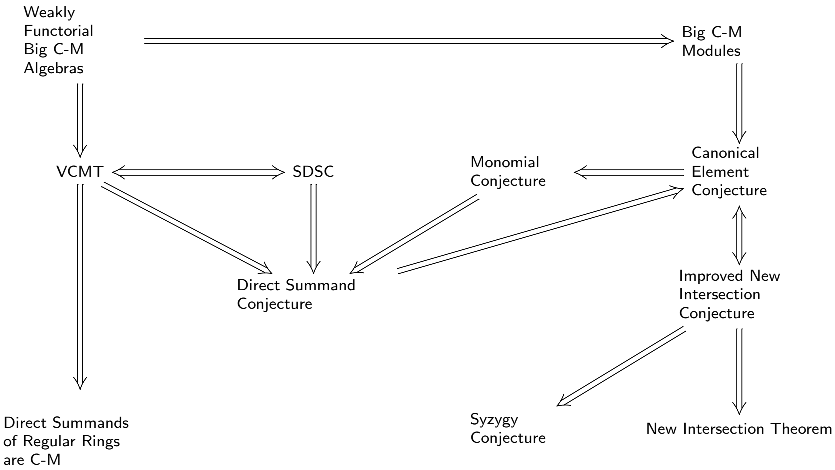
  3. True in equicharacteristic.

Homological Conjectures



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Note: Reductions change the rings R, A, so it is hard to carry special cases back and forth.

Short Proof of VCMT $\Rightarrow$ SDSC

- Want to show $xR \to Q$ splits. $(x \in \mathfrak{m}_R - \mathfrak{m}_R^2,\ \mathrm{ht}(Q) = 1,\ Q \supset xR)$
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- $\mathrm{H}_{\mathfrak{m}}^{d-1}(R + Q) \to \mathrm{H}_{\mathfrak{m}}^{d-1}(R/xR)$ is zero. $(d = \dim(R))$
- VCMT ⇒ DSC, so $\mathrm{H}_{\mathfrak{m}}^{d}(R) \to \mathrm{H}_{\mathfrak{m}}^{d}(R + Q)$ is injective.

Short Proof of VCMT ⇒ SDSC

$$\begin{array}{ccccccccc}
H^{d-1}_{\mathfrak{m}}(R+Q) & \xrightarrow{0} & H^{d-1}_{\mathfrak{m}}(R/xR) & \longrightarrow & H^{d}_{\mathfrak{m}}(Q) & \longrightarrow & H^{d}_{\mathfrak{m}}(R+Q) & \longrightarrow & 0 \\
\uparrow & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & H^{d-1}_{\mathfrak{m}}(R/xR) & \longrightarrow & H^{d}_{\mathfrak{m}}(xR) & \longrightarrow & H^{d}_{\mathfrak{m}}(R) & \longrightarrow & 0
\end{array}$$

Diagram Chase ⇒ $H^d(xR) \to H^d(Q)$ is injective
⇒ $xR \to Q$ splits.

Lemma

If we have a diagram with exact rows like:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \end{array}$$

Then the following are equivalent:

1. *There exist retractions such that the left diagram commutes.*

$$\begin{array}{ccc} N_1 & \longrightarrow & N_2 \\ \downarrow \rho_1 & & \downarrow \rho_2 \\ M_1 & \longrightarrow & M_2 \end{array} \qquad \begin{array}{ccc} N_2 & \longrightarrow & N_3 \\ \downarrow \rho_2 & & \downarrow \rho_3 \\ M_2 & \longrightarrow & M_3. \end{array}$$

2. *There exist retractions such that the right diagram commutes.*

Proposition (-)

If A is a domain, then the following are equivalent:

1. *There exist retractions $\rho : A \to R$ and $\eta : A/Q \to R/xR$ such that the following diagram commutes:*

$$\begin{array}{ccc} A & \longrightarrow & A/Q \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \eta} \\ R & \longrightarrow & R/xR. \end{array}$$

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2. *The map $xR \to Q$ splits.*

SDSC asserts that lower dimension splittings can be lifted in a compatible way.

Theorem (-)

Suppose $R/xR \hookrightarrow A/Q$ splits. If $\operatorname{Ext}^1_R(A, R) = 0$ or if x is a nonzerodivisor on $\operatorname{Ext}^1_R(A, R)$, then $xR \to Q$ splits.

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Theorem (-)

Suppose $\mathrm{Ext}^1_R(R + Q, R) = 0$ or x is a nonzerodivisor on $\mathrm{Ext}^1_R(R + Q, R)$. Then $xR \to Q$ splits.

Can drop the assumption on $R/xR \to A/Q$ splitting, since $(R + Q)/Q \cong R/xR$.

Theorem (-)

Let A be a domain, $\dim(A) = d$, and suppose DSC holds in dimension $d - 1$. Then SDSC holds in the following cases:

1. *A is Cohen-Macaulay.*
2. *A is an almost complete intersection domain.*
3. *A is normal and ω_A is S_3, where ω_A denotes the canonical module of A.*

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Theorem (-)

Suppose $A = R + Q$. Then $xR \to Q$ splits in each of the following cases:

1. *A is Cohen-Macaulay.*
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Theorem (-)

Let A be a complete local S_2 domain and let S be a complete intersection mapping onto A such that $\dim(S) = \dim(A)$. Let $\omega_A \subset S$ be the canonical module of A viewed as an ideal in S. If we assume DSC holds in lower dimension, then SDSC holds for S/ω_A.

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Theorem (-)

Let A be as above and let T be a complete intersection mapping onto $R + Q$ with $\dim(T) = \dim(R + Q)$. If the lift $\widetilde{Q} \subset T$ of Q contains ω_{R+Q}, then $xR \to Q$ splits.

Thank you!