1 Introduction

I study algebra, specifically commutative algebra. I am primarily interested in homological and computational properties of rings and modules. At it's core, commutative algebra is the study of systems of polynomial equations and the geometric objects they define. The subject is the foundation of algebraic geometry and algebraic number theory and I work on problems that have connections to both.

My recent projects concern two types of problems. In Section 2, I outline my work on some problems regarding bounding the projective dimension and Castelnuovo-Mumford regularity of homogeneous ideals in a polynomial ring. In Section 3, I outline my work on the Strong Direct Summand Conjecture, one of a set of open problems in the homological theory of modules over commutative rings. In Section 3.4, I briefly outline some of my computational work with the computer algebra system Macaulay2.

2 Bounding Projective Dimension and Regularity of Ideals

2.1 Background

Consider a polynomial ring over a field K, say $R = K[X_1, \dots, X_n]$. A free resolution of an R-module M is in some sense a measure of how close the R-module is to being free, that is, a direct sum of copies of R. Formally, a free resolution of M is an exact sequence of the form

$$0 \leftarrow M \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_{p-1} \leftarrow F_p \leftarrow 0,$$

where each module F_i is free. The Hilbert Syzygy Theorem guarantees that the minimal free resolution of M is finite and of length at most n. If we further assume that M is graded, we may take this to be a graded free resolution, in which case we write $F_i = R(-j)^{\beta_{ij}}$, where $R(-j)$ denotes a rank one free module with generator in degree j. The exponents $\beta_{ij} = \beta_{ij}(M)$ are called the **Betti numbers** of M and are invariants of M.

Two measures of the size and complexity of free resolution are **projective dimension** and **regularity**, which we now define. For a graded R-module M we set

$$\operatorname{pd} M = \max\{p \,|\, M \text{ has a graded free resolution of length } p\}$$

and

$$\operatorname{reg} M = \max\{k \,|\, \beta_{i,i+k}(M) \neq 0\}$$

(We remark that over R, finitely generated projective modules are free, hence we stick with the more common projective dimension rather than free dimension.) By the above statement, we know $\operatorname{pd} M \le n = \dim R$. The regularity of M is somewhat more difficult to bound as it measures the degree of the largest relation among the generators of M, or the relations upon those relations, etc.

It is sometimes convenient to only consider the Betti numbers of M rather than the full resolution. By convention we define the **Betti table** of M to be the integer matrix with $\beta_{i,i+j}$ in the ith column and jth row. (This takes into account that $\beta_{i,j} = 0$ if $j < i$.) We could now take as our definitions for $\operatorname{pd} M$ and $\operatorname{reg} M$ the indices of the last nonzero column and row, respectively.

Example 2.1. For example, if I is the homogeneous ideal $(v^3, w^3, xv^2 + yvw + zw^2)$ inside $R = K[v, w, x, y, z]$ and $M = R/I$, then M has graded free resolution:

$$0 \leftarrow R/I \leftarrow R \leftarrow R^3 \leftarrow R^8 \leftarrow R^{10} \leftarrow R^5 \leftarrow R \leftarrow 0$$

and Betti table

	0	1	2	3	4	5
total:	1	3	8	10	5	1
0:	1	.	.	.	.	.
1:	.	.	.	.	.	.
2:	.	3	.	.	.	.
3:	.	.	.	.	.	.
4:	.	.	8	10	5	1

By the remark above, $\operatorname{pd} R/I = 5$ and $\operatorname{reg} R/I = 4$.

2.2 Stillman's Question

We note that the Hilbert Syzygy Theorem guarantees that $\operatorname{pd}(R/I)$ is at most n, the number of variables. If we do not fix the number of variables, one can ask the following question, first posed by Stillman:

Question 1 (Stillman, [PS09, Problem 3.14])**.** Is there a bound, independent of n, on the projective dimension of ideals in $R = K[X_1, \ldots, X_n]$ which are generated by N homogeneous polynomials of given degrees $d_1, \ldots, d_N$?

Note that the number of variables is not fixed. This question is still open and only partial results are known. The same question where "projective dimension" is replaced with "regularity" was also posed by Stillman. It is equivalent to the above question by work of Caviglia. (See e.g. [Eng05].) In [Eng07], Engheta studied the case of 3 cubics and showed that the projective dimension of an ideal generated by 3 cubics is at most 36. He also published the first example of an ideal generated by 3 cubics with projective dimension 5 - the largest known example. Recently Ananyan and Hochster [AH11] proved a bound for ideals generated by forms of degree at most 2. Little else is known about this problem. An answer would have implications as to the computational complexity of computing Gröbner bases and resolutions.

Zhang [Zha11] conjectured that an upper bound for the projective dimension of an ideal I with N generators in degrees $d_1, \ldots, d_N$ is simply $\sum_{i=1}^{N} d_i$. This bound was suggested by his related work on local cohomology modules in characteristic $p > 0$. However, in [McC11] I showed that this was false by producing a family of ideals whose projective dimension grows exponentially if the number and degrees of the generators are allowed to grow linearly. One ideal in this family provided a simpler example of an ideal generated by 3 cubics with projective dimension 5. (See Example 2.1.)

In collaboration with J. Beder, L. Nunez, A. Seceleanu, B. Stone and B. Snapp [BMNB$^+$11], we produced yet another family of ideals whose projective dimension grows even faster. We proved that over any field there is a homogeneous ideal with only 3 generators in degree d with projective dimension at least $\sqrt{d}^{\sqrt{d}}$. Our most general theorem is the following:

Theorem 2.2. *Let K be a field. Fix $g \geq 2$ and integers $m_1, \dots, m_n$ such that $m_n \geq 0$, $m_{n-1} \geq 1$ and $m_i \geq 2$ for $1 \leq i \leq n-2$. Set:*

- $M_n = m_n$,
- $M_k = m_k - 1$ *for* $k < n$,
- $d_k = m_k + \cdots + m_n + 1$,
- $d = d_1$.

Finally, for $0 \leq k \leq n$ let

$$\mathcal{A}_k = \left\{ (a_{j,k'}) \;\middle|\; \begin{array}{l} 0 \leq a_{j,k'} \leq M_{k'} \text{ and } \sum_{j=1}^{g} a_{j,k'} = m_{k'} \text{ for} \\ 1 \leq k' \leq k, \text{ and } a_{j,k'} = 0 \text{ for } k < k' \leq n \end{array} \right\},$$

$$R = K[\mathsf{X}, y_{\mathsf{A}} \mid \mathsf{X} = (x_{j,k}), \mathsf{A} \in \mathcal{A}_n],$$

$$I_{g,(m_1,\dots,m_n)} = (x_{1,1}^d, \dots, x_{g,1}^d, f),$$

where

$$f = \sum_{k=1}^{n-1} \sum_{\mathsf{A} \in \mathcal{A}_{k-1}} \sum_{j=1}^{g} \mathsf{X}^{\mathsf{A}} x_{j,k}^{m_k} x_{j,k+1}^{d_{k+1}} + \sum_{\mathsf{B} \in \mathcal{A}_n} \mathsf{X}^{\mathsf{B}} y_{\mathsf{B}}.$$

Then

$$\operatorname{pd}(R/I) = gn + \left(\frac{(m_n + g - 1)!}{(g-1)!(m_n)!} \right) \prod_{i=1}^{n-1} \left(\frac{(m_i + g - 1)!}{(g-1)!(m_i)!} - g \right).$$

Hence any answer to Stillman's Question must be very large.

In the future there are many interesting questions I would like to address regarding this problem:

1. Exactly what are the regularities of the ideals defined above? Some partial answers are known and computational experiments indicate that the regularity grows very quickly but a general formula for the regularity is not known.
2. The Gröbner bases of these ideals are incredibly complicated. Can one write down a concise description of them?
3. Similarly can one say anything about the resolutions of these ideals?

2.3 Regularity Bounds

We retain the notation $R = K[X_1, \dots, X_n]$. More recently I have been working on bounding the regularity of cyclic modules and ideals in terms of some of the syzygies in the resolution. Let M be a graded R-module. Set

$$t_i(M) = \mathrm{reg}(\mathrm{Tor}_i(M, K)).$$

Then $t_i(M)$ is the maximal degree of an ith syzygy of M in its minimal graded free resolution. So we could define the regularity of M as

$$\mathrm{reg}(M) = \max_i \{t_i(M) - i \,|\, 0 \leq i \leq n\}.$$

It has been shown by Galligo, Giusti and Caviglia-Sbarra that

$$\mathrm{reg}(S/I) \leq (2t_1(R/I))^{2^{n-1}}.$$

See [Gal74], [Gal79], [Giu84], [CS05]. Examples of Mayer and Meyer [MM82] show that this doubly exponential behavior of regularity cannot be avoided. However, the large regularity occurs at the second syzygies of R/I - early in the resolution. The result below gives some explanation for this behavior.

In work still in progress, I have been able to show that one can bound the regularity of cyclic module R/I using as few as half of the numbers $t_i = t_i(R/I)$. Specifically, we have the following result:

Theorem 2.3. *Let $I \subset S$ be a homogeneous ideal. Set $h = \lceil n/2 \rceil$. Then*

$$\mathrm{reg}(R/I) \leq \sum_{i=1}^{h} t_i + \frac{\prod_{i=1}^{h} t_i}{(h-1)!}.$$

This bound is in general large, but not as large as the doubly exponential bound above. It shows that if doubly exponential syzygies do not occur in the first half of the resolution, they cannot occur in the latter half of the resolution of a cyclic module R/I. The proof involves a careful analysis of the numerics of the Boij-Söderberg decomposition of the Betti diagram of S/I. (See [ES09], [BS08b], [BS08a].)

3 Homological Conjectures

3.1 Background

The Homological Conjectures are a set of interconnected open problems that have their origins in the works of Auslander [Aus61], Bass [Bas63], Serre [Ser00], Peskine and Szpiro[PS09], and Hochster [Hoc73]. These problems have important ramifications in algebra and algebraic geometry including invariant theory, cohomology of vector bundles, the solutions of polynomial equations, characteristic-p techniques and intersection theory. In the following I will outline those problems that are connected with my research.

While there are more general versions, we focus on the local case and assume that all rings are Noetherian with unique maximal ideal. Most of the Homological Conjectures are known to be true in the case where the ring in question contains a field. Many of these conjectures remain open in the mixed characteristic case where the ring does not contain a field, though some partial results are known.

In [Hoc73], Hochster proved the following statement in the case where the ring R contains a field:

Conjecture 1 (The Direct Summand Conjecture)**.** *Let R be a regular local ring and A be a local ring which is finitely generated as an R-module. Then R is a direct summand of A.*

Much work has been done on the Direct Summand Conjecture (henceforth DSC), including Heitmann's proof [Hei02] of the DSC in dimension 3. Many people, including Hochster, Huneke, Dutta, Evans, Griffith, Koh, Goto and Roberts, have been instrumental in advancing progress on the Homological Conjecture. (See e.g. [Hoc83], [HH95], [Dut01], [Rob76], [EG81], [Got83], [Koh86]) In particular, Hochster reformulated the DSC into equivalent versions. One version, called the Canonical Element Conjecture, rephrases the DSC in terms of certain elements in local cohomology modules [Hoc83]. Another version, called the Monomial Conjecture, characterizes the DSC in terms of the nonvanishing of certain polynomial equations [Hoc73]. We discuss a stronger version of the Monomial Conjecture in the next section.

In more recent work, Hochster and Huneke [HH95] formulated a stronger conjecture:

Conjecture 2 (Vanishing Maps of Tor Conjecture(VMTC))**.** *Let $R \to A \to S$ be homomorphisms of Noetherian rings such that R and S are regular and such that A is module-finite over R. Let M be any R-module. Then the maps*

$$\operatorname{Tor}_i^R(M, A) \to \operatorname{Tor}_i^R(M, S)$$

are zero for all $i \geq 1$.

They showed that this conjecture is also known when the rings in question contain a field [HH95] and later Hochster proved the dimension 3 case [Hoc02], but the question is open in general. This new conjecture not only implies the DSC, but also implies that direct summands of regular rings are Cohen-Macaulay.

Later Ranganathan [Ran00] showed that the VMTC was actually equivalent to the following statement:

Conjecture 3 (Strong Direct Summand Conjecture)**.** *Let R be a regular local ring and let A be a module finite extension. Let Q be a height one prime ideal of A containing xR where x is a minimal generator of the maximal ideal of R. Then xR is a direct summand of Q.*

As the name suggests, the Strong Direct Summand Conjecture (SDSC) implies the DSC. Few of the partial results for the DSC have been extended to the SDSC. Moreover, the SDSC provides a new avenue for attacking the VMTC.

3.2 Results on the SDSC

In my thesis I proved the following special case of the SDSC:

Theorem 3.1 ([McC09])**.** *Suppose that* $\operatorname{Ext}^1_R(A,R)=0$ *or that* x *is a nonzerodivisor on* $\operatorname{Ext}^1_R(A,R)$*. Further suppose that* $R/xR\to A/Q$ *splits. Then the SDSC holds for* Q *and* A*; that is, the map* $xR\to Q$ *splits.*

We note that the condition that $\operatorname{Ext}^1_R(A,R)=0$ holds in three important cases.

Corollary 3.2. *Let* A *be a complete local domain of dimension* d*. Suppose the DSC holds for rings of dimension* $d-1$*. Then the SDSC holds in the following cases:*

1. A *is Cohen-Macaulay.*
2. A *is an almost complete intersection domain.*
3. A *is normal and* ω_A *is* S_3*, where* ω_A *denotes the canonical module of* A*.*

We may also assume that $A=R+Q$. In this case, the splitting of $R/xR\to A/Q$ is obvious, since, in this case, $A/Q=R/xR$. Thus we can state the following result without the inductive hypothesis of the previous corollary.

Theorem 3.3. *Let* $R\to A$ *be a module finite extension of complete local domains. Suppose* R *is regular. Let* x *be a minimal generator of the maximal ideal of* R *and let* Q *be a height one prime lying over* xR *in* A*. Set* $B=R+Q$*. Then the* $xR\to Q$ *splits in the following cases:*

1. B *is Cohen-Macaulay.*
2. B *is an almost complete intersection domain.*
3. B *is normal and* ω_B *is* S_3*, where* ω_B *denotes the canonical module of* B*.*

3.3 Strong Monomial Conjecture

In another part of my research [McC09], I have considered the following Strong Monomial Conjecture, as stated by Ranganathan in [Ran00]:

Conjecture 4 (Strong Monomial Conjecture(SMC))**.** *Let* A *be a local domain with system of parameters* $x_1,\ldots,x_d$*. Let* Q *be a height one prime of* A *containing* x_1*. Then for all* $t>0$*,*

$$x_1(x_1\cdots x_d)^t\notin(x_1^{t+1},\ldots,x_d^{t+1})Q.$$

The SMC implies the SDSC. The following is an extension of results by Dutta [Dut98] and Strooker and Stückrad [SS93] on the Monomial Conjecture.

Theorem 3.4. *The Strong Direct Summand Conjecture holds for all local rings if and only if it holds for all local almost complete intersection rings.*

Thus to prove the SMC, we need only prove it for this smaller class of rings.

Again, using the theorem above, I proved another equivalent form of the SMC in terms of Koszul homology modules. Keeping the notation above, we have the following:

Theorem 3.5. *The the Strong Monomial Conjecture holds for Q, z and $\mathbf{x}$ if and only if $\ell(J_z/\mathbf{x}J_z) > \ell(H_1(\mathbf{x}; S/J_z))$.*

Here $\ell(-)$ denotes the length of a module and $H_1(\mathbf{x}; S/J_z)$ denotes the first Koszul homology module on S/J_z with respect to the system of parameters $x_1, \ldots, x_d$. For an arbitrary module M, $H_1(\mathbf{x}; M)$ may not be finite length, but using another result of Dutta and Samuel's theory of superficial elements, I proved the following:

Theorem 3.6. *Suppose the residue field of A is infinite. Let $x_1, \ldots, x_d$ be a s.o.p. Let Q be a height one prime in A. Finally let J_{x_1} be as above. Then there exist $y_1, \ldots, y_{d-1} \in A$ such that $(x_1, \ldots, x_d) = (y_1, \ldots, y_{d-1}, x_d)$ and such that SMC holds for A, Q, $y_1, \ldots, y_{d-1}, x_d^t$ and x_1 for $s \gg 0$; that is,*

$$x_1(y_1 \cdots y_{d-1}, x_d^s)^t \notin (y_1^{t+1}, \ldots, y_{d-1}^{t+1}, (x_d^s)^{t+1})Q$$

Thus by altering the system of parameters slightly, we find cases where the SMC and hence SDSC holds.

Also recall that both of these conjectures imply the Direct Summand Conjecture, so any new special cases would immediately translate to new known cases of the Direct Summand Conjecture. For instance, in the case where the regular parameter $x = p$, the characteristic of the residue field of R, the direct summand conjecture is known for R/pR. Thus it may be possible to use characteristic p techniques for this case of the Strong Direct Summand Conjecture. These are two of the ideas I plan to explore in future research.

3.4 Computer Algebra

Macaulay2 [GS] is a computer algebra system developed by Daniel Grayson and Mike Stillman. It performs computations such as Gröbner bases and free resolutions useful to commutative algebraists and algebraic geometers. I have written several Macaulay2 packages which extend the functionality of the program. With Bart Snapp and Chris Cunningham, I created package to compute and perform algebra on power series defined via generating functions, rational functions or polynomials. It has the ability to maintain a given accuracy and dynamically adjust that accuracy if asked.

Related to the work in Section 2, my coauthors and I have developed a package to generate the ideals defined previously. This package gives us the ability to study the ideals and perhaps answer further questions about their structure. It is perhaps worth noting that the ideals in [McC09] and [BMNB$^+$11] were discovered with the help of many Macaulay2 computations.

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