DG homological algebra and solution to a question of Vasconcelos

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Assumption

\((R, m, k)\) is a local commutative noetherian ring with unity.
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Notation

\(\mathfrak{S}(R) = \{\text{isomorphism classes of semidualizing } R\text{-modules}\}.\)
Fact (Base-change)

If $R \to S$ is a local homomorphism of finite flat dimension, then $\mathcal{G}(R) \leftrightarrow \mathcal{G}(S)$ by $C \mapsto S \otimes_R C$. 
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Outline of proof

There is a flat local ring homomorphism $R \to (R', \mathfrak{m}R', \bar{k})$. Let $\mathbf{x} \in \mathfrak{m}R'$ be a maximal $R'$-sequence. Then $R'/\mathbf{x}R'$ is artinian and $\mathcal{S}(R) \hookrightarrow \mathcal{S}(R') \hookrightarrow \mathcal{S}(R'/\mathbf{x}R')$. A result of Happel shows that $\mathcal{S}(R'/\mathbf{x}R')$ is finite.

S. Nasseh and S. Sather-Wagstaff
A commutative differential graded (DG) $R$-algebra is

1. a graded commutative $R$-algebra $A = \bigoplus_{i=0}^{\infty} A_i$ with

2. a differential $\partial^A$ (i.e., a sequence of $R$-linear maps $\partial_i^A: A_i \to A_{i-1}$ such that $\partial_{i-1}^A \partial_i^A = 0$ for all $i$) such that $\partial^A$ satisfies the Leibniz Rule: for all $a_i \in A_i$ and $a_j \in A_j$

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Let $A$ be a DG $R$-algebra. A semi-free DG $A$-module $C$ is **semidualizing** if it is homologically finite and the natural map $A \rightarrow \text{Hom}_A(C, C)$ is a quasiisomorphism.
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Theorem (Nasseh, Sather-Wagstaff)

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1. There is a flat local ring homomorphism $R \to (R', mR', k)$ such that $R'$ is complete.

2. Let $x \in mR'$ be minimal generating sequence and $K = K^{R'}(x)$. Now, there exists a finite dimensional DG algebra $U$ over $\bar{k}$ such that

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3. This diagram and the lifting result imply that

$$\mathcal{G}_{dg}(R) \hookrightarrow \mathcal{G}_{dg}(R') \simeq \mathcal{G}_{dg}(K) \simeq \mathcal{G}_{dg}(U).$$
4. We parametrize the set of all DG $U$-modules with fixed underlying graded $\bar{k}$-vector space by an algebraic scheme.
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7. We prove that there can only be finitely many open orbits, so $\mathcal{S}_{dg}(U)$ is finite.