

Correction to “Simplicial monoids and Segal categories”

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In the paper [4], there is an error in the statement of Proposition 3.12. The given pair of maps is not in fact an adjoint pair, since the left adjoint does not preserve coproducts. Therefore, we give the following revised statement.

PROPOSITION 1. *There is a Quillen equivalence of model categories*

$$L : \mathcal{LSSets}_*^{\mathcal{T}_M} \rightleftarrows \mathcal{Alg}^{\mathcal{T}_M} : N.$$

The right adjoint functor N is just the forgetful functor; if we regard a \mathcal{T}_M -algebra as strictly local \mathcal{T}_M -diagram of simplicial sets, then this functor forgets the strictly local structure.

This proposition is then just a variation on Badzioch’s rigidification theorem [1].

We give a proof of this proposition, not just for the theory \mathcal{T}_M of monoids, but for any (possibly multi-sorted) algebraic theory, following the proof of the generalization of Badzioch’s rigidification theorem found in [3].

PROPOSITION 2. *Let \mathcal{T} be a (multi-sorted) algebraic theory. Then there is a Quillen equivalence of model categories between $\mathcal{Alg}^{\mathcal{T}}$ and $\mathcal{LSSets}_{\mathcal{O}}^{\mathcal{T}}$.*

The proof of the result that we would like to use is given by letting $\mathcal{T} = \mathcal{T}_M$, the theory of monoids, and letting $\mathcal{O} = *$, the set with a single element.

Here, if \mathcal{T} is an \mathcal{O} -sorted theory, $\mathcal{LSSets}_{\mathcal{O}}^{\mathcal{T}}$ denotes the category of functors $\mathcal{T} \rightarrow \mathcal{SSets}$ with the homotopy \mathcal{T} -algebra model structure, but with the additional condition that the image of the terminal object in \mathcal{T} is actually isomorphic to the constant simplicial set given by \mathcal{O} , rather than just weakly equivalent to it.

We need to find an adjoint pair of functors between $\mathcal{Alg}^{\mathcal{T}}$ and $\mathcal{LSSets}_{\mathcal{O}}^{\mathcal{T}}$ and prove that it is a Quillen equivalence. Let

$$J_{\mathcal{T}} : \mathcal{Alg}^{\mathcal{T}} \rightarrow \mathcal{SSets}_{\mathcal{O}}^{\mathcal{T}}$$

be the inclusion functor. We need to show we have an adjoint functor taking an arbitrary diagram in $\mathcal{SSets}^{\mathcal{T}}$ to a \mathcal{T} -algebra. Here, we use the idea that a \mathcal{T} -algebra is a strictly local diagram, as given by the following definition.

DEFINITION 3. Let \mathcal{D} be a small category and $\mathcal{SSets}^{\mathcal{D}}$ the category of functors $\mathcal{D} \rightarrow \mathcal{SSets}$. Let P be a set of morphisms in $\mathcal{SSets}^{\mathcal{D}}$. An object Y in $\mathcal{SSets}^{\mathcal{D}}$ is *strictly P -local* if for every morphism $f : A \rightarrow B$ in P , the induced map on function complexes

$$f^* : \text{Map}(B, Y) \rightarrow \text{Map}(A, Y)$$

is an isomorphism of simplicial sets. A map $g : C \rightarrow D$ in $\mathbb{S}Sets^{\mathcal{D}}$ is a *strict P -local equivalence* if for every strictly P -local object Y in $\mathbb{S}Sets^{\mathcal{D}}$, the induced map

$$g^* : \text{Map}(D, Y) \rightarrow \text{Map}(C, Y)$$

is an isomorphism of simplicial sets.

Now, given a category of \mathcal{D} -diagrams in $\mathbb{S}Sets$ and the full subcategory of strictly P -local diagrams for some set P of maps, we have the following result which can be proved just as in [3, 5.6].

LEMMA 4. *Consider two categories, the category of strictly local diagrams with respect to the set of maps $P = \{f : A \rightarrow B\}$, and the the category of diagrams $X : \mathcal{D} \rightarrow \mathbb{S}Sets$ which are strictly local with respect to only one of the maps in P . Then the forgetful functor from the first category to the second has a left adjoint.*

Now, we can apply this lemma using the fact that a strictly local \mathcal{T} -diagram is precisely a \mathcal{T} -algebra, and noting that the objects of $\mathcal{L}\mathbb{S}Sets_{\mathcal{O}}^{\mathcal{T}}$ are strictly local with respect to the map specifying the data on the image of the terminal object of \mathcal{T} .

Applying Lemma 4 to the functor $J_{\mathcal{T}}$, we obtain its left adjoint functor

$$K_{\mathcal{T}} : \mathbb{S}Sets_{\mathcal{O}}^{\mathcal{T}} \rightarrow \text{Alg}^{\mathcal{T}}.$$

Then the following proposition follows just as in the general case [3, 5.9].

PROPOSITION 5. *The adjoint pair of functors*

$$K_{\mathcal{T}} : \mathbb{S}Sets_{\mathcal{O}}^{\mathcal{T}} \rightleftarrows \text{Alg}^{\mathcal{T}} : J_{\mathcal{T}}.$$

is a Quillen pair.

Then, we can extend to the localized model structure $\mathcal{L}\mathbb{S}Sets_{\mathcal{O}}^{\mathcal{T}}$ just as in [3, 5.11].

PROPOSITION 6. *The adjoint pair*

$$K_{\mathcal{T}} : \mathcal{L}\mathbb{S}Sets^{\mathcal{T}} \rightleftarrows \text{Alg}^{\mathcal{T}} : J_{\mathcal{T}}$$

is a Quillen pair.

The only difference in the proof is that we can remove one map (the one with respect to which the objects of $\mathbb{S}Sets_{\mathcal{O}}^{\mathcal{T}}$ are already strictly local) as we localize to get $\mathcal{L}\mathbb{S}Sets_{\mathcal{O}}^{\mathcal{T}}$. With this minor change in the localization, the proof of Proposition 2 follows just as the proof of the general case [3, 5.13].

Then, the other correction that should be noted is that the model structure $\mathcal{L}\mathbb{S}Sets^{\mathcal{T}_M}$ should be removed from the chain of Quillen equivalences just preceding Section 5 of [4]. Although this chain is not explicitly given for the many-object case in Section 5, the same changes should be made there.

The author is grateful to the referee of [2] who pointed out this error.

References

- [1] Bernard Badzioch, Algebraic theories in homotopy theory, *Ann. of Math.* (2) 155 (2002), no. 3, 895–913.
- [2] J.E. Bergner, Adding inverses to diagrams encoding algebraic structures, preprint available at math.AT/0610291.
- [3] J.E. Bergner, Rigidification of algebras over multi-sorted theories, *Algebr. Geom. Topol.* 6 (2006) 1925-1955.

- [4] J.E. Bergner, Simplicial monoids and Segal categories, *Contemp. Math.* 431 (2007) 59-83.

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