COMBINATORIAL METHODS IN ALGEBRAIC TOPOLOGY

1. Geometric and abstract simplicial complexes

Let $v_0, v_1, \dots, v_k$ be points in $\mathbb{R}^n$. These points determine a hyperplane in $\mathbb{R}^n$, consisting of linear combinations

$$\lambda_0 v_0 + \lambda_1 v_1 + \cdots + \lambda_k v_k$$

with each $\lambda_i \in \mathbb{R}$ and

$$\sum_{i=0}^{k} \lambda_i = 1.$$

The same number of points may or may not determine a hyperplane of the same dimension. For example, consider three distinct points in $\mathbb{R}^2$. Then these points may determine all of $\mathbb{R}^2$, or they may just determine a line, namely, if all three of the points lie on the same line. How can we explain the difference between these two cases, more generally? In the case where all three points lie on the same line, they aren't "general" enough.

Definition 1.1. Points $v_0, v_1, \dots, v_k$ in $\mathbb{R}^n$ are in *general position* if any subset of them spans a strictly smaller hyperplane.

When three points all lie on the same line, then removing one of the points still results in the same line; we don't get a smaller hyperplane. However, if they three points don't all lie on the same line, then restricting to two of them gives a line, or a smaller hyperplane than all of $\mathbb{R}^2$. In this example, it was easy to explain what general position meant (whether the points were all on a line or not), but when we have more points in higher dimensions, it gets more complicated, so we have the formal definition of general position.

Notice that if $k > n$, then $v_0, \dots, v_k$ cannot be in general position in $\mathbb{R}^n$.

As you might guess, general position is closely related to the concept of linear independence. Recall from linear algebra that a set of points $\{v_1, \dots, v_k\}$ is *linearly independent* in $\mathbb{R}^n$ if, whenever

$$\sum_{i=1}^{k} c_i v_i = 0$$

for some $c_i \in \mathbb{R}$, it follows that $c_i = 0$ for all $1 \leq i \leq k$.

Exercise 1.2. Show that points $v_0, \dots, v_k$ are in general position if and only if $v_1 - v_0, v_2 - v_0, \dots, v_k - v_0$ are linearly independent.

You may prove this exercise directly, or you may find it useful to show that points in general position are *affinely independent*, in that if

$$\sum_{i=0}^{k} \lambda_i v_i = 0 \text{ and } \sum_{i=0}^{k} \lambda_i = 0$$

then $\lambda_i = 0$ for all $0 \leq \lambda_i \leq k$, and then prove the statement for $v_0, \dots, v_k$ affinely independent.

We want to consider shapes formed by points in general position, in a particular way.

Definition 1.3. A subset C of $\mathbb{R}^n$ is *convex* if for any points $p, q \in C$, the line segment connecting them is in C. In other words, for any $t \in [0,1]$, the point $tq + (1-t)p \in C$.

Exercise 1.4. Prove that the unit disk,

$$\{v \in \mathbb{R} \mid |v| \leq 1\}$$

is convex.

Given a set of points in $\mathbb{R}^n$, we want the smallest convex set containing those points, called the *convex hull*. Given points $v_0, \dots, v_k \in \mathbb{R}^n$, their convex hull is the set

$$\left\{ \sum_{i=0}^{k} \lambda_i v_i \mid \sum_{i=0}^{k} \lambda_i = 1, \lambda_i \in [0,1] \right\}.$$

In $\mathbb{R}^2$, you can imagine surrounding all the points with a loop of string and pulling the string tight; it will get caught on some or all of the points, and the area surrounded by this string is the convex hull. In $\mathbb{R}^3$, you could instead imagine throwing a sheet around the points and pulling it tight.

Now we consider the special kinds of convex hulls that we want to look at in this course.

Definition 1.5. Let $v_0, \dots, v_k$ be $k+1$ points in general position in $\mathbb{R}^n$, for $n > k$. The convex hull of these points is called a *k-simplex*, denoted Δ^n.

Since a 2-simplex is just a triangle, we often think of simplices (of any dimension) as "generalized triangles." The points $v_0, \dots, v_k$ are called *vertices* of a simplex.

Restricting to a subset of $v_0, \dots, v_k$, we get a simplex of a lower dimension, called a *face* of the original simplex. We have *face maps*, each sending a simplex to one of its faces of a lower dimension.

Simplices are nice convex spaces because they are completely determined by their vertices. However, we'd like to consider more general spaces as well, namely subspaces of $\mathbb{R}^n$ built from several simplices.

Definition 1.6. A *simplicial complex* K in $\mathbb{R}^n$ is a finite set of simplices $P_1, \dots, P_m$ such that:

(1) any face of P_i is also in K, and
(2) for any $1 \leq i, j \leq m$, $P_i \cap P_j$ is either a face of both P_i and P_j, or empty.

We can think of simplicial complexes as sets of simplices glued together in a nice way.

Many familiar objects can be thought of as simplicial complexes. For example, polyhedra made up of triangles are simplicial complexes, such as the boundaries of the tetrahedron, the octahedron, and the icosahedron. The solid tetrahedron is just a 3-simplex, and the solid versions of the others can be filled in with 3-simplices as well. Many other geometric shapes can be chopped up into triangles, so to speak, to be thought of as simplicial complexes.

So far, we've been thinking about simplices and simplicial complexes as geometric objects, as subsets of $\mathbb{R}^n$. But, as we've already noticed, simplices are completely determined by their vertices, and simplicial complexes by subsets of their vertices. In other words, we are really talking about something combinatorially defined.

Exercise 1.7. Write the simplicial complex (drawn on the board in class) combinatorially.

Definition 1.8. An *abstract simplicial complex* K is a pair (V_K, S_K), where V_K is a finite set (whose elements are called *vertices*) and S_K is a set of nonempty finite subsets of V_K (called *simplices*) such that all singleton subsets of V_K are in S_K, and, if $\sigma \in S_K$ and $\sigma' \subseteq \sigma$, then $\sigma' \subseteq S_K$.

Not only can we write a geometric simplicial complex as an abstract one, but we can *geometrically realize* an abstract one by assigning actual points in $\mathbb{R}^n$ to the elements of V_K. For small examples, we can just draw them. We denote the geometric realization of K by $|K|$.

Exercise 1.9. Draw a geometric simplicial complex $|K|$ associated to the abstract simplicial complex K given by:

$$V_K = \{a, b, c, d, e\}$$

$$S_K = \{\{a\}.\{b\}, \{c\}, \{d\}, \{e\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{c, e\}, \{d, e\},$$
$$\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{c, d, e\}, \{a, b, c, d\}\}.$$

However, in more complicated examples, with simplices in higher dimensions, to write it down formally we want a systematic approach. To start with, we want to be sure that n is sufficiently large, in particular at least as big as the largest set in S_K. To be on the safe side, we can make the following construction. Let $n = |V_K|$, and let e_i for $1 \leq i \leq n$ be the standard basis vectors in $\mathbb{R}^n$. Define a function $\varphi\colon V_K \to \{1, \dots n\}$ (assigning a number to each vertex). Then the geometric realization of K is given by

$$\bigcup_{\sigma \in S_K} c(\sigma)$$

where $c(\sigma)$ denotes the convex hull of the set

$$\{e_{\varphi(s)}\}_{s \in \sigma}.$$

Exercise 1.10. Given an abstract simplicial complex K with $|V_K| = n + 1$, what distinguishes K from the n-simplex Δ^n? (Here, since V_K is a set, $|V_K|$ denotes the cardinality of the set, not be confused with the geometric realization $|K|$ of a simplicial complex K.)

2. Topological spaces and their relationship with simplicial complexes

Recall that on the real line, we have notions of "open" and "closed" intervals. For example, $(0, 2)$ is open, $[3, 7]$ is closed, and $(-1, 3]$ is neither open nor closed. Consider the following properties of these kinds of intervals:

(1) The whole real line $\mathbb{R}$ can be considered to be open or closed.
(2) The union of any number of open intervals is still open.
(3) A finite intersection of open intervals is open.

Exercise 2.1. Give an example of an infinite intersection of open intervals that is no longer an open interval.

We can generalize this idea of "open" or "closed" subspaces to $\mathbb{R}^n$ in a couple of ways. One way is to think of an open set as the product of n open intervals in $\mathbb{R}$. This would give a kind of open box as an open set. Alternatively, we could think of open balls as open sets, where an open ball is the set of points less than some fixed distance from a given point. For example, we have the open unit disk

$$\{x \in \mathbb{R}^n \mid |x| < 1\}.$$

The closed unit disk would be

$$\{x \in \mathbb{R}^n \mid |x| \leq 1\}.$$

A more general open ball, with a different center and radius, would be given by

$$\{x \in \mathbb{R}^n \mid |x - a| < r, a \in \mathbb{R}^n, r \in \mathbb{R}, r > 0\}.$$

Generalizing these ideas leads to the idea of a topological space.

Definition 2.2. A *topology* on a set X is a collection of subsets of X called *open sets* satisfying the following properties:

(1) The whole set X is open, and so is the empty set ϕ.
(2) The finite intersection of open sets is open.
(3) Any union of open sets is open.

A set together with a topology is a *topological space.*

A subset of X is *closed* if it is the complement of an open set. In particular, both X and ϕ are always both open and closed. In general, a subset of X might be open, closed, both, or neither. (Think of intervals in $\mathbb{R}$.)

Example 2.3. The real line $\mathbb{R}$ with open sets the open intervals is a topological space. With either notion of open set given above, $\mathbb{R}^n$ is also a topological space.

Example 2.4. A very small example can be given as follows. Let $X = \{a, b, c\}$. A topology on X is given by the open sets $X, \phi, \{a\}, \{b\}, \{a, b\}$.

Exercise 2.5. Let X be a set with 5 elements. Give three examples of topologies on X that are distinctly different from one another. (So, don't just change the names of the points in one to get the others!)

Exercise 2.6. Find a different topology on the set of real numbers, other than the usual one already given.

Any subspace Y of a topological space X can be given the *subspace topology*, where a subset of Y is open in Y if and only if it is the intersection of Y with an open set of X. As we saw in the example in class, this definition can lead to some open sets in Y which are not open in X. Such examples again remind us that "open" doesn't always mean what we've intuitively always thought in $\mathbb{R}^n$. One way to think about how these strange examples arise is that, when Y becomes a topological space in its own right, it must be both open and closed. It may not have had one or both of these properties as a subset of X.

Thus, any geometric simplicial complex (regarded as a subspace of some $\mathbb{R}^n$) is a topological space. The main examples of topological spaces that we will consider are subsets of $\mathbb{R}^n$, but it is good to realize that there are a lot more out there, and that topological spaces can be very general, and very strange!

We can also combine spaces to get new topological spaces. For example, we can take the product $X \times Y$ of two topological spaces X and Y. If we consider open sets U of X and V of Y, then we can take $U \times V$ as open sets in $X \times Y$. These open sets don't actually form a topology, but they form a *basis* for a topology, in that if we take unions of them as open sets, we have a topology.

Exercise 2.7. Give an example of why the open sets $U \times V$ aren't enough for a topology on $X \times Y$.

Now, we'd like to think about what a function between topological spaces should be. Certainly they will be functions of sets, but they should preserve the topology in some way.

We often think of functions in this way in mathematics. In calculus, we deal with all kinds of functions $\mathbb{R} \to \mathbb{R}$, but often we restrict to continuous functions, or differentiable functions, depending on what we are doing. In linear algebra, we look at linear functions between vector spaces (those given by matrices). In group theory, we want functions between groups to preserve the operation, rather than just be maps of sets, so we consider group homomorphisms. We want to decide what the best notion of function is for topological spaces. We'll actually carry over some terminology from the example of $\mathbb{R}$.

Definition 2.8. A function of topological spaces $f\colon X \to Y$ is *continuous* if, for any open set U in Y, the set $f^{-1}(U)$ is open in X. These functions are often simply called *maps*.

Exercise 2.9. Check that if $f\colon \mathbb{R} \to R$ is a function on $\mathbb{R}$ with the usual topology, then this definition is equivalent to the usual $\delta - \varepsilon$ definition.

Yesterday we talked about abstract simplicial complexes and how to realize them geometrically. Today, we see that these geometric realizations are really topological spaces, when we think of them as subspaces of $\mathbb{R}^n$. But, we can also think of geometrically realizing functions of simplicial complexes to get continuous of topological spaces. But, what is the correct definition of functions of abstract simplicial complexes? A function $f\colon K \to L$ of simplicial complexes should be a pair of functions $V_K \to V_L$ and $S_K \to S_L$, which are each just maps of sets, and the second is compatible with the first, in that the singleton sets of S_K get sent to the appropriate singleton sets of S_L, and well-behaved with respect to the behavior of sub-simplices.

Then, a function $f\colon K \to L$ of abstract simplicial complexes induces a map $|f|\colon |K| \to |L|$ of topological spaces.

Recall from yesterday that we can think of simplicial complexes as a collection of simplices "glued together" in nice ways. When we glue, we identify some simplices together. We can also obtain topological spaces in this way. For example, we can identify the sides of a rectangle in various ways to obtain the torus T^2, the sphere S^2, the real projective plane $\mathbb{R}P^2$, and the Klein bottle K^2. We can identify only two of the sides to get the Möbius strip M^2. (See class notes for pictures of these spaces!)

If we "triangulate" the original rectangle, we can think of these spaces as geometric realizations of simplicial complexes. However, notice that these triangulations are not unique. Also notice that after we make identifications, we may not have a genuine, simplicial complex anymore. Still, for examples like T^2, it is much easier to visualize a triangulation of the square than a triangulation of T^2.

Exercise 2.10. Give two different triangulations of T^2 (or, rather, of the square in such a way that is compatible with the identifications), distinct from the ones given in class.

Exercise 2.11. Give a representation of $\mathbb{R}P^2$ as a rectangle with identifications on the sides.

Remark 2.12. You might wonder how the "gluing" process affects the topology. To be precise, we need to define the *quotient topology.* As with the subspace topology, for the purposes of this course, we just need to know that there is a canonical choice for the topology, and then not worry about it too much. Suppose that X is a topological space, and Y is obtained from X by identifying some points in X with one another. Then we can define a *quotient map* $q\colon X \to Y$, and then say that a subset U of Y is open if and only if $q^{-1}(U)$ is open in X. Notice that this definition guarantees that the quotient map q is continuous.

3. Homeomorphisms of topological spaces and invariants

We've talked about what maps of topological spaces should be, but we haven't yet talked about what it means for two topological spaces to be essentially the same. Certainly, they should be isomorphic as sets. But, there should be more structure. Going back to the example of groups, two groups are not necessarily isomorphic just because they have the same number of elements. They need to have an isomorphism of groups between them; in other words, we need to have a group homomorphism between them with an inverse which is also a homomorphism. We want a similar notion of "isomorphism" of topological spaces, although we usually use a different term.

Definition 3.1. A map $f\colon X \to Y$ is a *homeomorphism* if it is a continuous map with continuous inverse. In this case we say that X and Y are *homeomorphic*.

Notice that f is first an isomorphism of sets, since it needs to have an inverse for this definition to make sense.

Example 3.2. The real line $\mathbb{R}$ is homeomorphic to the open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ via the continuous function

$$\tan\colon (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}.$$

Example 3.3. We now give an example of an isomorphism of sets which is continuous but not a homeomorphism, in that its inverse is not continuous. Take the function $[0, 2\pi) \to S^1$ (where S^1 is the unit circle in $\mathbb{R}^2$) given by $t \mapsto (\cos t, \sin t)$. Notice that the inverse function is not continuous. (It is easiest to see using the $\delta - \varepsilon$ definition of continuity.)

Exercise 3.4. Prove that any open interval (a, b) is homeomorphic to $\mathbb{R}$.

Exercise 3.5. Describe a homeomorphism from $[0, 1) \times [0, 1) \to [0, 1] \times [0, 1)$. (You do NOT need to write it out explicitly! Just describe how it might work.)

In general, it is very difficult to determine whether two spaces are homeomorphic. Thus, it would be nice to have ways of showing that spaces are or are not homeomorphic using easier methods. One strategy is the use of invariants. An *invariant* is a property that is preserved under homeomorphism. So, that tells us that two spaces which are homeomorphic will share the same property. Now, two spaces that are not homeomorphic might also share this property, but we do know that if two spaces have different such properties, then they cannot be homeomorphic.

Another way to think about invariants is as follows. Given a space, assign to it some other mathematical object (such as a number, a group, a ring, etc.) such that if two spaces are homeomorphic, then they will be assigned to the same object.

The first invariant we might think of is cardinality. A topological space is first of all a set, and homeomorphisms must be isomorphisms of sets, so before checking whether two spaces are homeomorphic, we can see if they are isomorphic as sets. If they are not, then we know they can't possibly be homeomorphic. So, for example, any topology we put on a finite set will never give a space homeomorphic to $\mathbb{R}$. Similarly, any topology on $\mathbb{Z}$ will never be homeomorphic to $\mathbb{R}$.

However, most interesting spaces are (uncountably) infinite, so this isn't generally a very helpful invariant. The next invariant we'll look at will be more useful; it assigns a number to a space, and it uses simplicial complexes.

Let K be a simplicial complex, or a triangulated topological space, of dimension less than or equal to 2. (We could generalize to higher dimensions, but we won't.) Let v denote the number of vertices (0-simplices), e the number of "edges" (1-simplices), and f the number of "faces" (2-simplices). (Notice that this is a more specific definition of face than we have been using, but it is standard here.) Then the *Euler characteristic* of K is $\chi(K) = v - e + f$.

Example 3.6. Let K be the boundary of the 3-simplex. Then $\chi(K) = 4-6+4 = 2$.

You may have seen that $\chi(K) = 2$ for any convex polyhedron (not necessarily simplicial).

Exercise 3.7. Check that the Euler characteristic of the boundaries of the Platonic solids are all 2.

But, topologically all convex polyhedra are homeomorphic to the sphere, S^2. So, if the Euler characteristic is really an invariant, then it makes sense that we keep getting the same number. Furthermore, notice that the Euler characteristic is independent of the triangulation. If it were not, then it wouldn't be a topological invariant!

What about other spaces that we have seen? (Pictures given in class will be helpful here!)

Example 3.8. For the torus T^2, we get $\chi(T^2) = 1 - 3 + 2 = 0$.

In other words, T^2 is not homeomorphic to S^2.

Example 3.9. For the real projective plane, we get $\chi(\mathbb{R}P^2) = 2 - 3 + 2 = 1$.

Example 3.10. For the Klein bottle, we get $\chi(K^2) = 1 - 3 + 2 = 0$.

Thus, we now begin to see the limitations of our invariant. The Klein bottle is not homeomorphic to the torus, but they have the same Euler characteristic. So, we can't tell the difference between them in this way.

This example reminds us an important fact about invariants: they can tell us sometimes when spaces are different, but they can't tell us for sure if spaces are the same. So, if we get the same invariant (in this case, the same number), then can't conclude anything about whether the spaces are homeomorphic or not. For this reason, we need to have lots of invariants that tell us different things about spaces.

Many invariants are algebraic in nature. We find a way to assign a group or ring to a space in a way that preserves homeomorphism. One can think of algebraic topology as being the study of topological spaces using algebraic invariants.

4. Fundamental groups and homotopy

We are going to look at one such algebraic invariant, the fundamental group. (Other invariants are possible project topics.) We first need to understand the notion of homotopy.

Definition 4.1. Let $f, g \colon X \to Y$ be continuous maps of topological spaces. Then f is *homotopic* to g if there is a continuous map (called a *homotopy*) $F \colon X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for any $x \in X$. In this case we write $f \simeq g$.

The idea is that we get a family of functions, parameterized by the unit interval $I = [0, 1]$, continuously moving us from the function f to the function g.

Why should this be an important idea? We can think about topology as being a weakening of geometry, in that we don't care about the difference between two spaces if we can get one from the other by wiggling it a little. The idea behind a homotopy is that we also don't care about the difference between two maps if one can be thought of as being the other but wiggled a little.

Then we can ask how we could tell whether maps are homotopic or not. In some cases, the answer is simple.

Example 4.2. Let C be a convex subset of $\mathbb{R}^n$. Then, for any space X, any two maps $f, g \colon X \to C$ are homotopic. To find a homotopy, we can think of, for each point $x \in X$, connecting the images $f(x)$ and $g(x)$ by a straight line. For a general subspace of $\mathbb{R}^n$, this process won't work, since there is no guarantee that the line connecting the two points stays within the specified subset. But, since we assume that C is convex, we know by definition that it does. We call this homotopy the *straight-line* homotopy, and define it precisely as $F \colon X \times I \to C$ given by $(x, t) \mapsto (1 - t)f(x) + tg(x)$.

We'd like to know that homotopy is well-behaved, so it is good to check the following fact.

Exercise 4.3. Prove that homotopy defines an equivalence relation on the set of maps $X \to Y$, for any spaces X and Y. (In other words, so that it is reflexive, symmetric, and transitive.)

Once we know that homotopy defines an equivalence relation, then we can consider equivalence classes, or homotopy classes of maps between two spaces. Given a path f we denote its homotopy class $[f]$.

Another nice fact about homotopy is that it is well-behaved with respect to composition in the following sense. Suppose that $f \simeq g \colon X \to Y$, $h \colon Y \to Z$, and $k \colon W \to X$. Then $h \circ f \simeq h \circ g$, and $f \circ k \simeq g \circ k$.

An important example will be the case where X is itself the unit interval $I = [0, 1]$. A map $f \colon I \to Y$ just defines a *path* in Y. We want to consider homotopy classes of paths, in particular between paths that have the same endpoints. So, suppose we have two paths $f, g \colon I \to Y$ such that $f(0) = g(0)$ and $f(1) = g(1)$. If f and g are not homotopic, then we can think of them as being distinctly different ways of traveling between these two endpoints.

For example, if we take two points on the torus, we might guess that paths are non-homotopic if they wrap around each of the "holes" a different number of times. But, how could we tell for sure?

To approach this question, we first look at a process by which we can "compose" paths. Formally, this process is called *concatenation* of paths. If $f, g\colon I \to Y$ are paths such that $f(1) = g(0)$, then we can define $f \cdot g$ to be the path defined by

$$(f \cdot g)(t) = \begin{cases} f(2t) & 0 \le t \le \frac{1}{2} \\ g(2t-1) & \frac{1}{2} \le t \le 1. \end{cases}$$

This reparametrization allows us to retain our definition of a path as a continuous map from $I = [0,1]$ and not some other interval.

Given a path f, it has an *inverse path* f^{-1}, given by $f^{-1}(t) = f(1-t)$.

If we have two paths $f, g\colon I \to Y$ with the same endpoints, then we can take the concatenation of one path followed by the inverse of the other path. In doing so, we get a *loop*, or path $h\colon I \to Y$ with $h(0) = h(1)$. We can actually study homotopy classes of paths by understanding homotopy classes of loops in this way.

Suppose we fix a basepoint y_0 in our space Y and consider all loops based at y_0. In other words, we want to look at homotopy classes of paths $f\colon I \to Y$ with $f(0) = f(1) = y_0$. We have already seen that we can compose loops via concatenation; we extend this definition to a "multiplication" on homotopy classes of loops by defining $[f][g] = [f \cdot g]$.

Exercise 4.4. Show that this definition of multiplication is well-defined. In other words, show that it does not depend on which representative of the homotopy class you choose.

Notice that we have the homotopy class of the trivial loop $e\colon I \to Y$, given by $e(t) = y_0$ for all $t \in I$.

Exercise 4.5. Show that for any loop $f\colon I \to Y$, that $f \cdot f^{-1} \simeq e$.

Thus, if we define $[f]^{-1} = [f^{-1}]$, we have proved the following.

Theorem 4.6. *The set of homotopy classes of loops in Y with basepoint y_0 forms a group.*

This group is called the *fundamental group* of Y based at y_0, and is denoted $\pi_1(Y, y_0)$.

Example 4.7. Let C be a convex subset of $\mathbb{R}^n$. Since any two maps $X \to C$ are homotopic, we obtain that $\pi_1(C, c_0) = 0$, the trivial group, for any $c_0 \in C$.

We will often leave off the basepoint in the notation, for the following reason. If y_0 and y_1 are two choices of basepoint in Y which can be connected by a path, then it can be shown that $\pi_1(Y, y_0) \cong \pi_1(Y, y_1)$. A space where any two points can be connected by a path is called *path-connected*. We'll usually consider such spaces, so basepoints don't matter too much.

Example 4.8. It is difficult to prove, but not hard to believe, that $\pi_1(S^1) = \mathbb{Z}$. Think of a loop giving an integer based on how many times it winds around the circle and in which direction (clockwise or counterclockwise).

Another nice fact about the fundamental group is that it is well-behaved with respect to products.

Theorem 4.9. *For path-connected spaces X and Y, $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$*

Example 4.10. Notice that the torus T^2 is given by the product $S^1 \times S^1$. Thus, $\pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z}$.

Now suppose that we have two spaces X and Y, together with a continuous map $f\colon X \to Y$. Then we'd like to show that this map induces a homomorphism $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$. Let $[\alpha] \in \pi_1(X, x_0)$ be represented by a loop α. Then define $f_*([\alpha]) = [f \circ \alpha]$. Using the fact that homotopy is well-behaved with respect to composition, we can see that f_* is well-defined and a group homomorphism. Furthermore, if we have maps $f\colon X \to Y$ and $g\colon Y \to Z$, then $(g \circ f)_* = g_* \circ f_*$.

Now, we want to show that the fundamental group is in fact an invariant.

Theorem 4.11. *If $f\colon X \to Y$ is a homeomorphism, then $f_*\colon \pi_1(X) \to \pi_1(Y)$ is an isomorphism.*

Proof. Since f is a homeomorphism, we also have its inverse $f^{-1}\colon Y \to X$. Then $f_* \circ (f^{-1})_* = (\mathrm{id}_Y)_*$ and $(f^{-1})_* \circ f_* = (\mathrm{id}_X)_*$. But, it is not hard to see that $(\mathrm{id}_X)_*\colon \pi_1(X) \to \pi_1(X)$ is the identity isomorphism on $\pi_1(X)$. Thus, f_* is an isomorphism of groups with inverse $(f^{-1})_*$. □

The fundamental group is actually an invariant of a weaker notion of equivalence on topological spaces.

Definition 4.12. A map $f\colon X \to Y$ is a *homotopy equivalence* if there exists a map $g\colon Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. In this case we say that the spaces X and Y are *homotopy equivalent*.

A homotopy equivalence is weaker than a homeomorphism, in that we are asking that it only have an inverse only up to homotopy. A good exercise is to check that the above theorem is still true if we replace "homeomorphism" with "homotopy equivalence. Thus, we say that the fundamental group is *homotopy invariant*.

Why would we care about homotopy equivalence? Again, showing that spaces are homeomorphic is very hard; it is reasonable to look for a weaker condition to satisfy. Unfortunately, showing that spaces are homotopy equivalent is still very hard. However, we have many examples of spaces that are homotopy equivalent by not homeomorphic. In particular, while homeomorphism preserves dimension, homotopy equivalence may not.

Example 4.13. Let C be a convex subset of $\mathbb{R}^n$. Let $*$ denote the space with a single point, and let $c\colon C \to *$ be the collapse map and $i\colon * \to C$ be the inclusion of that point into C in any way. We can define a straight-line homotopy between id_C and $i \circ c$, showing that these maps are homotopy equivalences. Thus, C is homotopy equivalent to a point. Such a space is called *contractible*.

Example 4.14. Let $f\colon S^1 \to \mathbb{R}^2\backslash\{0\}$ be the inclusion map. Define $g\colon \mathbb{R}^2\backslash\{0\} \to S^1$ by

$$g(x) = \frac{x}{|x|}.$$

Then $g \circ f = \mathrm{id}_{S^1}$, and $f \circ g \simeq \mathrm{id}_{\mathbb{R}^2\backslash\{0\}}$ via the homotopy $G\colon \mathbb{R}^2\backslash\{0\} \times I \to \mathbb{R}^2\backslash\{0\}$ given by

$$(x, t) \mapsto (1 - t)x + t\left(\frac{x}{|x|}\right).$$

Thus, f is a homotopy equivalence, and the unit sphere S^1 is homotopy equivalent to the punctured plane.

To finish our discussion of fundamental groups, we'd like to look at why simplicial complexes are useful in helping to compute them. First, we need the idea of free groups.

Suppose we want to build up a group from a single element in the simplest possible way, without making any unnecessary identifications. Let's call this element a, and we want to assume that we have some other element e which will be the identity element, so we assume that $ae = ea = a$ and $ee = e$. If we want to get a group, we need a to have an inverse element, say a^{-1}, so $aa^{-1} = a^{-1}a = e$. But we can multiply a with itself to get a new element aa which we write a^2. This element has to have an inverse, and the group axioms force it to be the same as $a^{-1}a^{-1} = a^{-2}$. Then we can keep combining elements to get a group $\langle a \rangle$ which has elements $\dots a^{-2}, a^{-1}, e, a, a^2, \dots$. It is not hard to see that $\langle a \rangle \cong Z$, sending each element to its exponent. This group is called the *free group on one generator*.

We now, suppose we start off with two non-identity elements a and b. Then of course we get all powers of a as before, plus all powers of b, but we also get words like ab and ba and $b^{-2}a4$ and $ba^{-5}b^{-8}a^{15}$ and so forth. We call these elements *words* in the letters a, b, a^{-1}, and b^{-1}. These words also form a group; we can multiply two words by concatenation; for example, $ab \cdot ba^{-1} = ab^2a^{-1}$. Sometimes we need to simplify after concatenation, such as for $ba^2 \cdot a^{-3}ba = ba^{-1}ba$. This group is denoted $\langle a, b \rangle$ and is the *free group on two generators*. Notice that it is not abelian, because we have not identified ab with ba. It is sometimes written as $\mathbb{Z} * \mathbb{Z}$, the *free product* of $\mathbb{Z}$ with itself.

What is the difference between $\langle a, b \rangle \cong \mathbb{Z} * \mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}$? The second group is abelian, whereas the first is not. We could get it by adding the relation $ab = ba$. Given any free group, we can get another group by imposing some kind of relation for when two elements should be identified. Using the description we have just given, we could write $\mathbb{Z} \times \mathbb{Z}$ as $\langle a, b \mid ab = ba \rangle$. The a and b (on the left side) are the *generators* and on the right we have the relation $ab = ba$. A description of a group by generators and relations is called a *presentation*. There are many possible presentations for a given group, and it is very difficult in general to tell if two presentations give the same group, but we won't get into that problem here.

Example 4.15. The cyclic group $\mathbb{Z}/n$ has presentation $\langle a \mid a^n = e \rangle$.

We'd like to look at some topological spaces for which we can get a nice presentation for their fundamental groups. Basically we want to consider topological spaces obtained by gluing the sides of a polygon together. The surfaces we constructed last time will provide our examples. The following is a rough description.

Let P be a polygon with edges labeled by letters $a_1, \dots, a_n$ and directions, and let X be the space obtained by identifying the sides of P which have the same label, according to the assigned directions. Then a presentation for X can be given as follows. Pick a vertex of P and go around the edges in a consistent direction, reading off the letters, and either leaving them alone (if you are following the assigned direction) or taking their inverse (if you are going against the assigned direction). This process gives a word in $a_1, \dots, a_n$ and their inverses; call it w. Then $\pi_1(X) \cong \langle a_1, \dots a_n \mid w = e \rangle$.

Example 4.16. Although we already know what it is, let's use this method to compute $\pi_1(T^2)$. Calling the sides either a or b, we get a presentation

$$\langle a, b \mid ab^{-1}a^{-1}b = e \rangle.$$

Rearranging the letters, we get the same relation as we did previously for $\mathbb{Z} \times \mathbb{Z}$.

Example 4.17. A presentation of $\pi_1(\mathbb{R}P^2)$ is $\langle a \mid a^2 = e\rangle$. Notice that this group is just $\mathbb{Z}/2$.

Exercise 4.18. Use this method to find $\pi_1(K^2)$.

In class, I made the mistake of asking you to try this for S^2, which doesn't work. The problem is that this process requires that you only have one vertex after identifications are made. Remember that we showed that if we take two circles stuck together at one point, $S^1 \vee S^1$, then its fundamental group is the free group on one generator, and more generally for several circles stuck together. The idea is that, if you ignore the inside of the polygon for a moment, you want to get this bunch of circles after identifying the relations. Then, filling in the inside of the polygon and making identifications gives us the relations in the fundamental group. In the way we built S^2, we still had two vertices, so we can't compute its fundamental group this way.

5. From simplicial complexes to simplicial sets

So far, we've seen that (abstract) simplicial complexes and topological spaces are closely related to one another in a variety of ways. Given an abstract simplicial complex, we can take its geometric realization to get a topological space. At least in some nice cases, we can triangulate topological spaces to get simplicial complexes. Doing so helps us to compute invariants on the space such as the Euler characteristic or the fundamental group. Today, we are going to shift our perspective in two ways. First, from now on we are going to consider spaces up to homotopy equivalence, not up to homeomorphism. In other words, we are going to be doing "homotopy theory." Second, instead of looking at why simplicial complexes are good for understanding topological spaces, we are going to investigate why they aren't good enough. Our goal is to find some kind of combinatorial objects that model topological spaces, and, as we will see, we are going to need something more general than simplicial complexes. In doing so, you'll get an idea of how some modern homotopy theorists think about things.

We've looked at products of spaces and products of groups. We can also define products of abstract simplicial complexes. Suppose K and L are simplicial complexes. A natural way to define their product $K \times L$ is as follows. Let $V_{K \times L} = V_K \times V_L$, so the vertices of the product are just pairs of vertices of the original simplicial complexes. What about simplices? First, notice that we have two projection maps, $p_1 \colon V_K \times V_L \to V_K$, given by $(v, v') \mapsto v$, and $p_2 \colon V_K \times V_L \to V_L$, given by $(v, v') \mapsto v'$. Now, we define

$$S_{K \times L} = \{\text{all subsets } \sigma \text{ of } V_{K \times L} \text{ with } p_1(\sigma) \in S_K, p_2(\sigma) \in S_L\}.$$

Example 5.1. Let Δ^1 be the 1-simplex and $\partial\Delta^1$ be its boundary, which is just two points. Thus, $V_{\Delta^1} = \{v_0, v_1\}$ and $V_{\partial\Delta^1} = \{v_0', v_1'\}$, with

$$S_{\Delta^1} = \{\{v_0\}, \{v_1\}, \{v_0, v_1\}\}$$

and

$$S_{\partial\Delta^1} = \{\{v_0'\}, \{v_1\}\}.$$

Using our definition of product, we get

$$V_{\Delta^1 \times \partial\Delta^1} = \{v_0, v_1, v_0', v_1'\}$$

and

$$S_{\Delta^1 \times \partial\Delta^1} = \{\{(v_0, v_0')\}, \{(v_0, v_1')\}, \{(v_1, v_0')\}, \{(v_1, v_1')\}, \{(v_0, v_0'), (v_1, v_0')\}, \{(v_0, v_1'), (v_1, v_1')\}\}.$$

Thus, the geometric realization $|\Delta^1 \times \partial\Delta^1|$ looks as we would expect it to: two disjoint line segments.

However, it is not always the case that the geometric realization of a simplicial complex is the same as the product of the geometric realizations!

Example 5.2. Consider $\Delta^1 \times \Delta^1$. Using notation from above, we get

$$V_{\Delta^1 \times \Delta^1} = \{(v_0, v_0), (v_0, v_1), (v_1, v_0), (v_1, v_1)\}.$$

We won't write them all out here, but the elements of $S_{\Delta^1 \times \Delta^1}$ are all the subsets of $V_{\Delta^1 \times \Delta^1}$. In particular, $\Delta^1 \times \Delta^1$ is a 3-simplex! So, its geometric realization is a tetrahedron. However, $|\Delta^1| \times |\Delta^1|$ is a square, something 2-dimensional.

Exercise 5.3. Check that $S_{\Delta^1 \times \Delta^1}$ really is as claimed in the example.

So, geometric realization and products do not play well together. We'd like to modify our definition of simplicial complex so that products are better behaved.

Definition 5.4. An *oriented* simplicial complex is a simplicial complex with a partial ordering on its vertices.

Given any abstract simplicial complex, we can make it an oriented simplicial complex by ordering its vertices. Then each 1-simplex has a "direction" given by the ordering. We now have the notion of an oriented n-simplex, which we'll now denote $\Delta[n]$. We can geometrically realize an oriented simplicial complex just as before; we just don't use the orientation.

Now, given oriented simplicial complexes K and L, we define their product $K \times L$ to have vertices $V_{K\times L} = V_K \times V_L$. (Note that this set is still partially ordered.) Then the simplices are given by

$$S_{K\times L} = \{\text{totally ordered subsets } \sigma \text{ of } V_{K\times L} \text{ with } p_1(\sigma) \in S_K, p_2(\sigma) \in S_L\}.$$

Let's try our problematic example from before.

Example 5.5. Let's look at $\Delta[1] \times \Delta[1]$. We label the vertices of $\Delta[1]$ by v_0, v_1 as before, and order them $v_0 < v_1$. Then the set of vertices is the same as before. However, now we do not get all possible subsets, because two of our vertices, (v_0, v_1) and (v_1, v_0), are incomparable. The result is that we get an oriented simplicial complex whose geometric realization is just a square, just as we'd like.

Exercise 5.6. Write out the elements of $S_{\Delta[1]\times\Delta[1]}$ and show that it gives an oriented simplicial complex whose geometric realization is $|\Delta[1]| \times |\Delta[1]|$.

Exercise 5.7. Work out the vertex set and simplex set for $\Delta[1] \times \Delta[2]$ and show that its geometric realization is what you expect.

So, by orienting our simplicial complexes we get objects that better represent topological spaces. However, they still aren't quite the objects that we want. We've been using one other construction on spaces that doesn't carry over to ordered simplicial complexes, namely gluing. For example, we can take an interval and glue its ends together to get a circle. What if we tried to do this on the level of (oriented) simplicial complexes? We'd start with $\Delta[1]$, and then identify its vertices. But, then we'd only have one vertex, which means we'd have to have a 0-simplex, since simplices are completely determined by their vertices. To get a "circle" we'd need to take the boundary of the 2-simplex, which has 3 vertices and 3 1-simplices. There doesn't seem to be a natural way to get such an object via gluing the ends of a 1-simplex.

So, what we want to do now is abandon the condition that the simplices in a simplicial complex have to be determined by their vertices. In other words, we want to consider "simplices" which don't necessarily look like triangles. So, in the situation just described, we could have a 1-simplex which starts and ends at the same vertex, looking like a loop. There are two flavors of such combinatorial objects, Δ-sets and simplicial sets.

Let $[n]$ denote the ordered set with $n+1$ elements: $\{0 < 1 < \cdots < n\}$. Taking all possible integers $n \geq 0$, we can look at order-preserving maps between $[n]$ and $[m]$. Let's try some simple examples.

There are two possible maps $[0] \to [1]$, given by $0 \mapsto 0$ and $0 \mapsto 1$. There is only one map $[1] \to [0]$, since $[0]$ only has one element. If we consider maps $[2] \to [1]$, there are several possibilities. We could take $0, 1 \mapsto 0$ and $2 \mapsto 1$, or $0 \mapsto 0$ and $1, 2 \mapsto 1$. We could also send all elements of [2] to a single element of [1], but such a map could be obtained by composing with maps between [0] and [1], so we'll ignore them; we really want to construct generating maps between these ordered sets. Not counting constant maps, we get three maps $[1] \to [2]$: $0 \mapsto 0$ and $1 \mapsto 1$, $0 \mapsto 1$ and $1 \mapsto 2$, and $0 \mapsto 0$ and $1 \mapsto 2$.

Thus, we could put all these maps together into a diagram, with $n+1$ generating maps $[n-1] \to [n]$ and $n+1$ generating maps $[n+1] \to [n]$. Notice that we have some relations between these maps. This diagram looks like

$$[0] \rightleftarrows [1] \rightleftarrows [2] \rightleftarrows \cdots .$$

Now, if we replace $[n]$ with the n-simplex, the upward maps look a lot like face maps, except that they are going in the wrong direction; they look more like the inclusion of a face, than picking a face off a higher simplex. While it may seem like a strange thing to do, let's take the arrows in this diagram and turn them around. Now, we can think of the downward maps as face maps.

$$[0] \leftrightarrows [1] \leftrightarrows [2] \leftrightarrows \cdots .$$

What are the new upward maps? Well, they don't really have a counterpart in simplicial complexes as we've thought of them, but we can get an idea this way. Imagine regarding a 0-simplex as a "degenerate" 1-simplex, or as an even more degenerate 2-simplex. Or think of a 1-simplex as the one and only side of a 2-simplex that doesn't look much like a 2-simplex as we've thought of it, because two of its sides are collapsed in some way. That is what these upward maps are doing, sending a lower-dimensional "simplex" to the set of higher-dimensional ones so that it can be regarded as living there, too. Some people want to have these degeneracy maps around; other people don't.

Definition 5.8. A Δ-*set* X is a collection of sets $X_0, X_1, X_2, \dots$ together with face maps $d_i \colon X_n \to X_{n+1}$ for $0 \leq i \leq n$, satisfying $d_i d_j = d_{j-1} d_i$ for $i < j$.

You can check that this relation comes from the original relations in our diagram of ordered sets.

Definition 5.9. A *simplicial set* X is a collection of sets $X_0, X_1, X_2, \dots$ together with face maps $d_i \colon X_n \to X_{n-1}$ for $0 \leq i \leq n$ and degeneracy maps $s_i \colon X_n \to X_{n+1}$ for $0 \leq i \leq n$, satisfying several relations.

Thus, a simplicial set looks like a diagram

$$X_0 \leftrightarrows X_1 \leftrightarrows X_2 \leftrightarrows \cdots .$$

Now, $\Delta[1]$ as a Δ-set looks the same as it does as an ordered simplicial complex, but as a simplicial set, it has degenerate simplices of arbitrarily high dimensions! In fact, because of these degeneracies, simplicial sets are always infinite-dimensional.

Furthermore, notice that Δ-sets and simplicial sets are not subject to the constraints that we have on simplicial complexes, as described previously. We still have some trouble collapsing higher-dimensional simplices if we use Δ-sets; simplicial sets are useful there, due to the degeneracies.

Fact 5.10. Simplicial sets are combinatorial models for spaces.

What does this mean? While it is difficult to explain precisely at this stage, they have the same "homotopy theory." In other words, if you only care about spaces up to homotopy equivalence, then it is just as good to consider simplicial sets up to "homotopy" also, where "equivalence" of simplicial sets is defined to be homotopy equivalence after geometric realization. Some homotopy theorists (like me) take this fact so far as to call simplicial sets "spaces" and rarely actually think about topological spaces at all; we find it more convenient to do most things in terms of simplicial sets instead. While they are difficult to understand at first, they end up being extremely useful.

6. Nerves, categories, and functors

We'd like to look at a particular construction for taking a particular kind of simplicial set and getting a new one. Since it is easier to visualize, we'll start with examples using ordered simplicial complexes instead.

Let K be an ordered simplicial set of dimension 1. It is really just a *directed graph* or *quiver*, although they can be more general. (Again, by assuming it forms a simplicial complex we are making restrictions on where we can have 1-simplices.) We can build a possibly higher-dimensional simplicial complex using the *nerve* construction, as follows. Whenever we have a configuration of 1-simplices $\cdot \to \cdot \to \cdot$, we want to add another 1-simplex and fill it in with a 2-simplex. Whenever we have three such 1-simplices in order, $\cdot \to \cdot \to \cdot \to \cdot$, we fill in the 2-simplices as given by the previous step, and then add in another 1-simplex, plus the necessary 2-simplices which we can then fill in with a 3-simplex. More generally, whenever we have a string of n 1-simplices, with compatible ordering in this way, we fill it in with an n-simplex.

As you can see from a few examples, this process can get complicated pretty quickly. If we don't restrict to simplicial complexes, but do the same process with 1-dimensional Δ-sets, or simplicial sets with everything degenerate above level 1, we can do the same thing, but it can get even more complicated very quickly. While this can be a useful construction in its own right, it should remind you of a familiar process, composition of functions.

Let's look at the diagram $\cdot \to \cdot \to \cdot$ again. Since we are using arrows, you might be led to think of functions. But, what are they functions between? Well, so far, just dots, but maybe the dots represent vector spaces or groups or topological spaces or whatever. But, if we do that, then functions compose, and we should get a third arrow, going from the first dot to the second dot. And, we want to think of going along that arrow as the same as going along the other two arrows. So, filling it in with a 2-simplex indicates that the two ways of going across the triangle are the same. If we take longer strings of arrows, then we should get more composites, and we shade in more simplices to indicate that processes are the same.

Example 6.1. Let's look at an example which is not a simplicial complex, but a more general Δ-set: a single vertex v and a single 1-simplex g starting and ending at v. What would the nerve of this graph look like? We could compose g with itself, and then compose the composite with g again, and so forth. So, even in this example we get an infinite-dimensional Δ-set! If we call it K, then K_n consists of the n-fold composite of g with itself. The face maps just eliminate one of the factors.

Now, let's think about K as a simplicial set. What do the degeneracy maps give us? Now, the vertex v sitting in K_0 is also regarded as a degenerate element of K_1, a kind of "identity map" for v. Then degenerate elements of the sets K_n are n-fold composites where some of the factors are given by v rather than g.

You might start to think of K as a group here, but it isn't, as there is no "inverse" for g. But, we have composition which is associative, and an identity element, so K is a *monoid*.

Now, if we look back at some of our previous examples of simplicial complexes, if we regard them as simplicial sets instead, we get degeneracies of each vertex which can again be thought of as "identity maps." Unlike in the example just given, not all arrows compose with one another, because there is more than one vertex, and composition doesn't make sense unless arrows match up along the vertices in the correct way. But, we get a structure that looks like a "monoid with many vertices."

Definition 6.2. A *category* $\mathcal{C}$ is a set (or class) of *objects* together with:

(1) for any ordered pair (a, b) of objects, a set $\mathrm{Hom}(a, b)$ of *morphisms* (or arrows) $f : a \to b$, and
(2) for any ordered triple (a, b, c) of objects, a function $\mathrm{Hom}(a, b)\times\mathrm{Hom}(b, c) \to \mathrm{Hom}(a, c)$, giving composites of morphisms, say $(f, g) \mapsto g \circ f$, satisfying:
   - composition is associative, so, if we have
   $$a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d,$$
   then $h \circ (g \circ f) = (h \circ g) \circ f$, and
   - for each object b there is an identity morphism $\mathrm{id}_b \in \mathrm{Hom}(b, b)$ such that, if $f : a \to b$, then $\mathrm{id}_b \circ f = f$, and, if $g : b \to c$, $g \circ \mathrm{id}_b = g$.

Example 6.3. Modifying what we've done previously, we get a category $\mathcal{C}$ given by:

$$\begin{tikzcd} & b \arrow[dr, "g"] & \\ a \arrow[ur, "f"] \arrow[rr, "g\circ f"] & & c. \end{tikzcd}$$

The set of objects is $\mathrm{ob}(\mathcal{C}) = \{a, b, c\}$, and the morphism sets are $\mathrm{Hom}(a, b) = \{f\}$, $\mathrm{Hom}(b, c) = \{g\}$, $\mathrm{Hom}(a, c) = \{g \circ f\}$, $\mathrm{Hom}(a, a) = \{\mathrm{id}_a\}$, $\mathrm{Hom}(b, b) = \{\mathrm{id}_b\}$, $\mathrm{Hom}(c, c) = \{\mathrm{id}_c\}$, and $\mathrm{Hom}(b, a) = \mathrm{Hom}(c, b) = \mathrm{Hom}(c, a) = \phi$.

Example 6.4. An even simpler example with no composites is given by $\mathrm{ob}(\mathcal{C}) = \{a, b, c\}$, $\mathrm{Hom}(b, a) = \{f\}$, $\mathrm{Hom}(b, c) = \{g\}$, with just the identity maps from objects to themselves, and all other hom-sets empty.

Example 6.5. Any monoid is a category with one object. A group is also a category with one object, with the property that any morphism has an inverse. We call such morphisms *isomorphisms*. Any category for which all morphisms are isomorphisms is called a *groupoid* and can be thought of as a generalization of a group, just as a category is a generalization of a monoid.

Example 6.6. A partially ordered set can be thought of as a category, with objects the elements of the set and a single arrow $a \to b$ if $a < b$, and no other non-identity morphisms.

Example 6.7. This language gives us a new way to describe the collection of finite ordered sets $[n]$ for $n \geq 0$ and order-preserving morphisms between them. This category is called $\mathbf{\Delta}$. Recall that we formally turned all the arrows in this category around to describe simplicial sets; this category is called $\mathbf{\Delta}^{op}$. In general, given a category $\mathcal{C}$ has an *opposite category* $\mathcal{C}^{op}$ with the same objects as $\mathcal{C}$ but with all the arrows turned around.

Now, I correct something I said at the beginning of the section. We don't really want to take nerves of some kinds of simplicial complexes or simplicial sets (although we can); we really want to take the nerves of categories. Now, the composites are already there, so we don't have to add them in. We just fill in the 2-simplices to indicate when particular arrow is the composite of two others, and analogously for higher-dimensional simplices. Then, we can get a space from a category by taking the geometric realization of its nerve.

The examples we've seen so far are "small" categories. They are small in that we can essentially write them down, but they are also *small* in a technical sense, in that they have a set of objects. There are many examples of categories, many of which are familiar to you, even if you haven't thought about them as such, which are "large" in that they have a proper class (something larger than a set) of objects.

Why do we make the distinction? Recall Russell's Paradox, which led to the need for a good definition of set: the problematic notion of the set of all sets that do not contain themselves. It follows that we don't think of the collection of all sets as forming a "set" but something bigger, which we call a *class*, or a *proper class* to indicate that it is not actually a set. There is really no problem with thinking of the objects of a category as forming a class rather than a set. We just want the hom-sets ($\mathrm{Hom}(a, b)$ for any pair a, b of objects) to form a set, rather than a proper class.

Example 6.8. Since the functions between two sets forms a set, we have the category $\mathcal{S}ets$ of sets and functions between them.

Example 6.9. There is a category $\mathcal{V}ect$ of vector spaces and linear maps between them.

Example 6.10. There is a category $\mathcal{G}p$ of groups and group homomorphisms. Similarly, we have a category $\mathcal{M}on$ of monoids and monoid homomorphisms, and $\mathcal{R}ing$ of rings and ring homomorphisms.

Example 6.11. We have been studying the category $\mathcal{T}op$ of topological spaces and continuous maps between them, as well as the category $\mathcal{ASC}$ of abstract simplicial complexes and functions between them. We also have the category $\mathcal{SS}ets$ of simplicial sets and functions between them.

So, we can see here that our ongoing theme of trying to find appropriate functions for mathematical objects has been useful as then regarding those objects as living in some category. But, what about categories themselves? Can we have a "function" from one category to another? What would one look like? We'd want objects to get sent to objects, and morphisms to get sent to morphisms, but it would have to be done in such a way that everything is compatible.

Definition 6.12. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F\colon \mathcal{C} \to \mathcal{D}$ is a pair of functions, both of which are still denoted F:

- $F\colon \mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$, so an object a of $\mathcal{C}$ goes to an object $F(a)$ in $\mathcal{D}$, and
- $F\colon$ morphisms$(\mathcal{C}) \to$ morphisms$(\mathcal{D})$, which can be described as itself a collection of functions, one for each pair (a, b) of objects in $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(a, b) \to \mathrm{Hom}_{\mathcal{D}}(F(a), F(b))$, where $f\colon a \to b$ gets sent to $F(f)\colon F(a) \to F(b)$

such that $F(\mathrm{id}_a) = \mathrm{id}_{F(a)}$ for any $a \in \mathrm{ob}(\mathcal{C})$, and, for any pair of morphisms (f, g) such that $g \circ f$ exists, $F(g \circ f) = F(g) \circ F(f)$.

We've actually seen lots and lots of examples of functors in this class, even if we haven't called them such.

Example 6.13. The fundamental group defines a functor $\pi_1\colon \mathcal{T}op \to \mathcal{G}p$, since for any space X we get a group $\pi_1(X)$, and for any map of spaces $f\colon X \to Y$, we get a group homomorphism $F(f) = f_*\colon \pi_1(X) \to \pi_1(Y)$.

Example 6.14. Geometric realization defines a functor $|-|\colon \mathcal{SS}ets \to \mathcal{T}op$. (We could replace the category of simplicial sets with the category of simplicial complexes, or the category of ordered simplicial complexes, or the category of Δ-sets.

Example 6.15. Recall the definition of a simplicial set. The really slick way to define a simplicial set X is as a functor $X\colon \mathbf{\Delta}^{op} \to \mathcal{S}ets$. Every object $[n]$ gets assigned to some set X_n. Every morphism in $\mathbf{\Delta}^{op}$ gets assigned to a map of sets, and the relations that hold in $\mathbf{\Delta}^{op}$ will still be required to hold for the maps of sets.

Example 6.16. There is a (large) category $\mathcal{C}at$ whose objects are (small) categories and whose morphisms are functors between them. Then the nerve construction defines a functor $N\colon \mathcal{C}at \to \mathcal{SS}ets$.

We could also restrict to a subcategory of $\mathcal{C}at$, say $\mathcal{G}p$. Then we get a functor $N\colon \mathcal{G}p \to \mathcal{SS}ets$. We can also compose with geometric realization to get a functor $B\colon \mathcal{G}p \to \mathcal{T}op$. This functor is called the *classifying space* functor. It has some very nice properties; for example, given a group G, $\pi(BG) = G$.

I want to conclude with some discussion of what the language of categories and functors is good for. It can be somewhat daunting at first, but it is a very good framework in which to think about the relationships between different kinds of mathematical objects. For example, really nice invariants of topological spaces are functorial, in that they define functors; it is nice if our invariant not only tells us about a given space, but about a map of spaces. So, we get information not just about individual spaces, but about relationships between spaces. The Euler characteristic is not functorial, whereas the fundamental group is.

It is also key to think about categories and functors as a *language* that unifies concepts that are recurring all over mathematics. Sometimes a theorem can be proved in more generality, using the language of categories, so you don't have to keep proving it all over again in each new situation. For this reason, categories are used all over mathematics, especially in topology and algebra.