REEDY CATEGORIES AND THE Θ-CONSTRUCTION

JULIA E. BERGNER AND CHARLES REZK

Abstract. We use the notion of multi-Reedy category to prove that, if $\mathcal{C}$ is a Reedy category, then $\Theta\mathcal{C}$ is also a Reedy category. This result gives a new proof that the categories Θ_n are Reedy categories. We then define elegant Reedy categories, for which we prove that the Reedy and injective model structures coincide.

1. Introduction

In this note, we generalize two known facts about the category Δ, which has the structure of a Reedy category. The first is that the categories Θ_k, obtained from Δ via iterations of the Θ construction, are also Reedy categories. The second is that, on the category of simplicial presheaves on Δ, or functors $\Delta^{op} \to \mathcal{SS}ets$, the Reedy and injective model structures agree.

For the first generalization, we use the notion of multi-Reedy category to prove that for any Reedy category $\mathcal{C}$, we get that $\Theta\mathcal{C}$ is also a Reedy category. For the second, we give a sufficient condition for the Reedy and injective model structures to coincide; such a Reedy category we call *elegant*.

A Reedy category is defined by two subcategories, the direct and inverse subcategories, and a degree function. (A precise definition is given in Section 2.) A consequence of the results of this paper is that the Reedy structure on Θ_k is characterized by:

1. A map $\alpha\colon \theta \to \theta'$ is in Θ_k^- if and only if $F\alpha\colon F\theta \to F\theta'$ is an epimorphism in $\mathrm{Psh}(\Theta_k)$.
2. A map $\alpha\colon \theta \to \theta'$ is in Θ_k^+ if and only if $F\alpha\colon F\theta \to F\theta'$ is a monomorphism in $\mathrm{Psh}(\Theta_k)$.
3. There is a degree function $\deg\colon \mathrm{ob}(\Theta_k) \to \mathbb{N}$, defined inductively by
$$\deg([m](\theta_1, \ldots, \theta_m)) = m + \sum_{i=1}^{m} \deg(\theta_i).$$

Here, $\mathrm{Psh}(\Theta_k)$ denotes the category of presheaves on Θ_k and F denotes the Yoneda functor. In itself, this result is not new; Θ_k was shown to be a Reedy category by Berger [2].

Terminology 1.1. We note two differences in terms from other work. First, by "multicategory" we mean a generalization of a category in which a function has a single input but possibly multiple (or no) outputs. This notion is dual to the usual definition of multicategory, in which a function has multiple inputs but a single output, equivalently defined as a colored operad. Perhaps the structure we use would better be called a co-multicategory, but we do not because it would further complicate already cumbersome terminology.

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Second, some of the ideas in this work are related to similar ones used by Berger and Moerdijk in [3]. For example, their definition of EZ-category is more general than ours, in that some of their examples fit into their framework of generalized Reedy categories.

2. Reedy categories and multi-Reedy categories

2.1. **Presheaf categories.** Given a small category $\mathcal{C}$, we write $\mathrm{Psh}(\mathcal{C})$ for the category of functors $\mathcal{C}^{op} \to \mathrm{Set}$. We write $\mathrm{Psh}(\mathcal{C}, \mathcal{M})$ for the category of functors $\mathcal{C}^{op} \to \mathcal{M}$, where $\mathcal{M}$ is any category.

We write $F_{\mathcal{C}} \colon \mathcal{C} \to \mathrm{Psh}(\mathcal{C})$ for the Yoneda functor, defined by $(F_{\mathcal{C}} c)(d) = \mathcal{C}(d, c)$. When clear from the context, we will usually write F for $F_{\mathcal{C}}$.

We use the following terminology. Given an object c of $\mathcal{C}$ and a presheaf $X \colon \mathcal{C} \to \mathcal{S}et$, a *c-point* of X is an element of the set $X(c)$. Given an c-point $x \in X(c)$, we write $\bar{x} \colon Fc \to X$ for the map which classifies the element in $X(c)$.

2.2. **Reedy categories.** Recall that a *Reedy category* is a small category $\mathcal{C}$ equipped with two wide subcategories (i.e., subcategories with all objects of $\mathcal{C}$), denoted $\mathcal{C}^+$ and $\mathcal{C}^-$ and called the *direct* and *inverse* subcategories, respectively, together with a *degree function* $\deg \colon \mathrm{ob}(\mathcal{C}) \to \mathbb{N}$ such that the following hold.

1. Every morphism α in $\mathcal{C}$ admits a unique factorization of the form $\alpha = \alpha^+ \alpha^-$, where α^+ is in $\mathcal{C}^+$ and α^- is in $\mathcal{C}^-$.
2. For every morphism $\alpha \colon c \to d$ in $\mathcal{C}^+$ we have $\deg(c) \leq \deg(d)$, and for every morphism $\alpha \colon c \to d$ in $\mathcal{C}^-$, we have $\deg(c) \geq \deg(d)$. In either case, equality holds if and only if α is an identity map.

Note that, as a consequence, $\mathcal{C}^+ \cap \mathcal{C}^-$ consists exactly of the identity maps of all the objects, and that identity maps are the only isomorphisms in $\mathcal{C}$. Furthermore, for all objects c of $\mathcal{C}$, the slice categories $(c \downarrow \mathcal{C}^-)$ and $(\mathcal{C}^+ \downarrow c)$ have finite-dimensional nerve.

2.3. **Multi-Reedy categories.** Associated to a Reedy category is a structure which looks much like that of a multicategory, which has morphisms with one input object but possibly multiple output objects.

Let $\mathcal{C}$ be a small category. For any finite sequence of objects $c, d_1, \ldots, d_m$ in $\mathcal{C}$, with $m \geq 0$, define

$$\mathcal{C}(c; d_1, \ldots, d_m) \overset{\text{def}}{=} \mathcal{C}(c, d_1) \times \cdots \times \mathcal{C}(c, d_m).$$

This notation also extends to empty sequences; $\mathcal{C}(c;)$ denotes a one-point set. We refer to elements $\alpha = (\alpha_s \colon c \to d_s)_{s=1,\ldots,m}$ as *multimorphisms* of $\mathcal{C}$, and we sometimes use the notation $\alpha \colon c \to d_1, \ldots, d_m$ for such a multimorphism. Let $\mathcal{C}(*)$ denote the *symmetric multicategory* whose objects are those of $\mathcal{C}$, and whose multimorphisms $c \to d_1, \ldots, d_s$ are as indicated above. Note that $\mathcal{C}(c; d) = \mathcal{C}(c, d)$, and that $\mathcal{C}$ may be viewed as a subcategory of the multicategory $\mathcal{C}(*)$.

Definition 2.4. A *multi-Reedy category* is a small category $\mathcal{C}$ equipped with a wide subcategory $\mathcal{C}^- \subseteq \mathcal{C}$, and a wide sub-multicategory $\mathcal{C}^+(*) \subseteq \mathcal{C}(*)$, together with a function $\deg \colon \mathrm{ob}(\mathcal{C}) \to \mathbb{N}$ such that the following hold:

(1) Every multimorphism
$$\alpha = (\alpha_s \colon c \to d_s)_{s=1,\dots,m}$$
in $\mathcal{C}(*)$ admits a unique factorization of the form $\alpha = \alpha^+\alpha^-$, where $\alpha^- \colon c \to x$ is a morphism in $\mathcal{C}^-$ and $\alpha^+ \colon x \to d_1, \dots, d_m$ is a multimorphism in $\mathcal{C}^+(*)$.

(2) For every multimorphism $\alpha \colon c \to d_1, \dots, d_m$ in $\mathcal{C}^+(*)$ we have
$$\deg(c) \leq \sum_{i=1}^{m} \deg(d_i).$$
If $\alpha \colon c \to d$ is a morphism in $\mathcal{C}^+ = \mathcal{C} \cap \mathcal{C}^+(*)$, then $\deg(c) = \deg(d)$ if and only if α is an identity map. For every morphism $\alpha \colon c \to d$ in $\mathcal{C}^-$, we have $\deg(c) \geq \deg(d)$, with equality if and only if α is an identity map.

Note that for degree reasons, $\mathcal{C}^+(c;) = \varnothing$ if $\deg(c) > 0$, while $\mathcal{C}^+(c;)$ is non-empty if $\deg(c) = 0$. In particular, if c is any object in $\mathcal{C}$, there exists a unique map $\sigma \colon c \to c_0$ in $\mathcal{C}^-$ where c_0 is an object of degree 0.

The proof of the following proposition follows from the above constructions.

Proposition 2.5. *If $\mathcal{C}$ is a multi-Reedy category, then $\mathcal{C}$ is a Reedy category with inverse category $\mathcal{C}^-$, direct category $\mathcal{C}^+ = \mathcal{C}^+(*) \cap \mathcal{C}$, and degree function* deg.

Example 2.6. The terminal category $\mathcal{C} = 1$, with the subcategory $\mathcal{C}^- = \mathcal{C}$ and the submulticategory $\mathcal{C}^+(*) = \mathcal{C}(*)$, and degree function $\deg \colon \mathrm{ob}(\mathcal{C}) \to \mathbb{N}$, defined by $d(1) = 0$, is a multi-Reedy category.

Example 2.7. Let Δ be the skeletal category of non-empty finite totally ordered sets. Let $\Delta^- \subseteq \Delta$ be the subcategory of Δ consisting of surjective maps, and let $\Delta^+(*) \subseteq \Delta(*)$ be the submulticategory consisting of sequences of maps
$$\alpha_s \colon [m] \to [n_s], \qquad s = 1, \dots, u$$
which form a monomorphic family; i.e., if $\beta, \beta' \colon [k] \to [m]$ satisfy $\alpha_s\beta = \alpha_s\beta'$ for all $s = 1, \dots, u$, then $\beta = \beta'$. Let $\deg \colon \mathrm{ob}(\Delta) \to \mathbb{N}$ be defined by $\deg([m]) = m$. Then Δ is a multi-Reedy category.

Note that the set $\Delta^+([m]; [n_1], \dots, [n_r])$ corresponds to the set of non-degenerate m-simplices in the prism $\Delta^{n_1} \times \cdots \times \Delta^{n_r}$.

Remark 2.8. Note that the notion of multi-Reedy category, while having the structure of a multicategory, is being associated to an ordinary Reedy category. This definition can be extended to an arbitrary multicategory, thus giving rise to the notion of a "Reedy multicategory", as we investigate briefly in Section 5.

2.9. **The Θ construction.** Given a small category $\mathcal{C}$, we define $\Theta\mathcal{C}$ to be the category whose objects are $[m](c_1, \dots, c_m)$ where $m \geq 0$, and $c_1, \dots, c_m \in \mathrm{ob}(\mathcal{C})$, and such that morphisms
$$[m](c_1, \dots, c_m) \to [n](d_1, \dots, d_n)$$
correspond to $(\alpha, \{f_i\})$, where $\alpha \colon [m] \to [n]$ is a morphism of Δ, and for each $i = 1, \dots, m$,
$$f_i \colon c_i \to d_{\delta(i-1)+1}, \cdots, d_{\alpha(i)}$$
is a multimorphism in $\mathcal{C}(*)$, which is to say $f_i = (f_{ij})$ where $f_{ij} \colon c_i \to d_j$ for $\delta(i-1) < j \leq \delta(i)$ is a morphism of $\mathcal{C}$.

2.10. **$\Theta\mathcal{C}$ is a multi-Reedy category whenever $\mathcal{C}$ is.** Let $\mathcal{C}$ be a multi-Reedy category, and consider the category $\Theta\mathcal{C}$. We make the following definitions.

- Let $(\Theta\mathcal{C})^- \subseteq \Theta\mathcal{C}$ be the collection of morphisms
$$f = (\alpha, \{f_i\})\colon [m](c_1, \dots, c_m) \to [n](d_1, \dots, d_n)$$
such that $\alpha\colon [m] \to [n]$ is in Δ^-, and for each $i = 1, \dots, m$ such that $\alpha(i-1) < \alpha(i)$, the map $f_i\colon c_i \to d_{\alpha(i)}$ is in $\mathcal{C}^-$.
- Let $(\Theta\mathcal{C})^+(*) \subseteq (\Theta\mathcal{C})(*)$ be the collection of multimorphisms $f = (f_s)_{s=1,\dots,u}$, where
$$f_s = (\alpha_s, \{f_{si}\})\colon [m](c_1, \dots, c_m) \to [n_s](d_{s1}, \dots, d_{sn_s})$$
such that the multimap $\alpha = (\alpha_s)\colon [m] \to [n_1], \dots, [n_u]$ is in $\Delta^+(*)$ and for each i, the multimap
$$(f_{sij}\colon c_i \to d_{sj})_{s=1,\dots,u,\ j=\alpha_s(i-1)+1,\dots,\alpha_s(i)}$$
is in $\mathcal{C}^+(*)$.
- Let $\deg\colon \mathrm{ob}(\Theta\mathcal{C}) \to \mathbb{N}$ be defined by
$$\deg([m](c_1, \dots, c_m)) = m + \sum_{i=1}^{m} \deg(c_i).$$

Proposition 2.11. *Let $\mathcal{C}$ be a multi-Reedy category. Then $\Theta\mathcal{C}$ is a multi-Reedy category, with $(\Theta\mathcal{C})^-$, $(\Theta\mathcal{C})^+(*)$ and $\deg\colon \mathrm{ob}(\Theta\mathcal{C}) \to \mathbb{N}$ defined as above. In particular, $\Theta\mathcal{C}$ admits the structure of a Reedy category.*

We assure the reader that the proof is entirely formal; however, we will do our best to obscure the point by presenting a proof full of tedious multiple subscripts.

Proof. First we observe that $(\Theta\mathcal{C})^-$ is closed under composition and contains identity maps; i.e., it is a subcategory of $\Theta\mathcal{C}$. Notice that $(\Theta\mathcal{C})^-$ contains all identity maps. Suppose we have two morphisms in $(\Theta\mathcal{C})^-$ of the form
$$[m](c_1, \dots, c_m) \xrightarrow{f=(\sigma,f_i)} [n](d_1, \dots, d_n) \xrightarrow{g=(\tau,g_j)} [p](e_1, \dots, e_p).$$
The composite has the form $h = (\tau\sigma, h_i)$, where h_i is defined exactly if $\tau\sigma(i-1) < \tau\sigma(i)$, in which case $h_i = g_{\sigma(i)} f_i\colon c_i \to e_{\tau\sigma(i)}$. Since $\tau\sigma \in \Delta^-$ and $h_i = g_{\sigma(i)} f_i$ is in $\mathcal{C}^-$, we see that h is in $(\Theta\mathcal{C})^-$.

Next we observe that $(\Theta\mathcal{C})^+(*)$ is closed under multi-composition and contains identity maps; i.e., it is a sub-multicategory of $(\Theta\mathcal{C})(*)$. Again, note that $(\Theta\mathcal{C})^+(*)$ contains all identity maps. Suppose we have a multimorphism f in $(\Theta\mathcal{C})^+(*)$ of the form
$$f = \big(f_s\colon [m](c_1, \dots, c_m) \to [n_s](d_{s1}, \dots, d_{sn_s})\big)_{s=1,\dots,u}$$
with $f_s = (\delta_s, f_{si})$ where $\delta_s\colon [m] \to [n_s]$ and $f_{si} = (f_{sij}\colon c_i \to d_j)_{\delta_s(i-1)<j\leq\delta_s(i)}$, and suppose we have a sequence of multimorphisms $g_1, \dots, g_u$ in $(\Theta\mathcal{C})^-(*)$, with each g_s of the form
$$g_s = \big(g_{st}\colon [n_s](d_{s1}, \dots, d_{sn_s}) \to [p_{st}](e_{st1}, \dots, e_{stp_{st}})\big)_{t=1,\dots,v_s},$$
where $g_{st} = (\varepsilon_{st}, g_{stj})$, with $\varepsilon_{st}\colon [n_s] \to [p_{st}]$ and $g_{stj} = (g_{stjk}\colon d_{sj} \to e_{stk})_{\varepsilon_{st}(j-1)<k\leq\varepsilon_{st}(j)}$. The composite multimorphism
$$h = \big(h_{st}\colon [m](c_1, \dots, c_m) \to [p_{st}](e_{st1}, \dots, e_{stp_{st}})\big)_{s=1,\dots,u,\ t=1,\dots,v_u}$$

is such that for each $s = 1, \dots, u$ and $t = 1, \dots, v_s$, the map $h_{st} = (\varepsilon_{st}\delta_s, h_{sti})$ in $\Theta\mathcal{C}$ is defined so that the multimap h_{sti} in $\mathcal{C}(*)$ is given by

$$h_{sti} = \big(h_{stijk} = g_{stjk}f_{sij} \colon c_i \to e_{stk}\big)_{\delta_s(i-1)<j\leq\delta_s(i),\ \varepsilon_{st}(j-1)<k\leq\varepsilon_{st}(j)}.$$

Since $\Delta^+(*)$ is a sub-multicategory of $\Delta(*)$, we get that that $(\varepsilon_{st}\delta_s \colon [n] \to [p_{st}])_{st}$ is a multimorphism in $\Delta^+(*)$, while since $\mathcal{C}^+(*)$ is a sub-multicategory of $\mathcal{C}(*)$, we have that for each s, t, and i, the multimap h_{sti} is in $\mathcal{C}^+(*)$. Thus, the multimap h is in $(\Theta\mathcal{C})^+(*)$ as desired.

Next, suppose we are given a multimorphism $f = (f_s)_{s=1,\dots,u}$ in $(\Theta\mathcal{C})(*)$, where

$$f_s = (\alpha_s, f_{si}) \colon [m](c_1, \dots, c_m) \to [p_s](e_{s1}, \dots, e_{sp_s}).$$

We will show that there is a unique factorization of f into a morphism g of $(\Theta\mathcal{C})^-$ followed by a multimorphism h of $(\Theta\mathcal{C})^+(*)$. Since $\alpha = (\alpha_s)_{s=1,\dots,u}$ is a multimorphism in $\Delta(*)$ it admits a *unique* factorization $\alpha = \delta\sigma$, where $\sigma \colon [m] \to [n]$ is in Δ^- and

$$\delta = (\delta_s \colon [n] \to [p_s])_{s=1,\dots,u}$$

is in $\Delta^+(*)$. Thus, any factorization $f = hg$ of the kind we want must be such that

$$g = (\sigma, g_i), \qquad g_i \colon c_i \to d_{\sigma(i)} \text{ defined when } \sigma(i-1) < \sigma(i),$$

and $h = (h_s)_{s=1,\dots,u}$ such that

$$h_s = (\delta_s, h_{sj}), \qquad h_{sj} \colon d_j \to e_{\delta(j-1)+1}, \dots, e_{\delta(j)},$$

and so that for each $i = 1, \dots, m$ such that $\sigma(i-1) < \sigma(i)$, the composite of the morphism g_i of $\mathcal{C}$ with the multimorphism $h_{*\sigma(i)} = (h_{s\sigma(i)})_{s=1,\dots,u}$ of $\mathcal{C}(*)$ must be equal to the multimorphism $f_{*i} = (f_{si})_{s=1,\dots,m}$ of $\mathcal{C}(*)$. In fact, since $\mathcal{C}$ is a multi-Reedy category, there is a *unique* way to produce a factorization $f_{*i} = h_{*\sigma(i)}g_i$ with the property that g_i is in $\mathcal{C}^-$ and $h_{*\sigma(i)}$ is in $\mathcal{C}^+(*)$.

Suppose that $f = (\sigma, f_i) \colon [m](c_1, \dots, c_m) \to [n](d_1, \dots, d_n)$ is a morphism in $(\Theta\mathcal{C})^-$. Then

$$\begin{aligned}
\deg([m](c_1, \dots, c_m)) &= m + \sum_{i=1}^{m} \deg(c_i) \\
&\geq n + \sum_{j=1}^{n} \deg(d_j) \\
&= \deg([n](d_1, \dots, d_n)).
\end{aligned}$$

The inequality in the second line follows from the fact that $m \geq n$ since $\sigma \in \Delta^-$, and the fact that for each $j = 1, \dots, n$, there is exactly one i such that $\sigma(i-1) < j \leq \sigma(i)$, for which the map $f_i \colon c_i \to d_j$ in $\mathcal{C}^-$, whence $\deg(c_i) \geq \deg(d_j)$.

If equality of degrees hold, then we must have $m = n$, whence σ is the identity map of $[m]$, and thus we must have $\deg(c_i) = \deg(d_i)$ for all $i = 1, \dots, m$, whence each f_i is the identity map of c_i.

Suppose that $f = (f_s)_{s=1,\dots,u}$ is a multimorphism in $(\Theta\mathcal{C})^+(*)$, where

$$f_s = (\delta_s, f_{si}) \colon [m](c_1, \dots, c_m) \to [n_s](d_{s1}, \dots, d_{sn_s}).$$

Since $(\delta_s) \in \Delta^+(*)$, we have $m \leq \sum_{s=1}^u n_s$. For each $i = 1, \ldots, m$, the multimorphism

$$f_{*i*} = \big(f_{sij} \colon c_i \to d_{sj}\big)_{s=1,\ldots,u,\; j=\delta_s(i-1)+1,\ldots,\delta_s(i)}$$

is in $\mathcal{C}^+(*)$, and thus

$$\deg(c_i) \leq \sum_{s=1}^{u} \sum_{j=\delta_s(i-1)+1}^{\delta_s(i)} \deg(d_{sj}).$$

For each $s = 1, \ldots, u$ and $j = 1, \ldots, n_s$, there is *at most* one i such that $\delta_s(i-1) < j \leq \delta_s(i)$. Thus

$$\sum_{i=1}^{m} \deg(c_i) \leq \sum_{i=1}^{m} \sum_{s=1}^{u} \sum_{j=\delta_s(i-1)+1}^{\delta_s(i)} \deg(d_{sj}) \leq \sum_{s=1}^{u} \sum_{j=1}^{n_s} \deg(d_{sj}),$$

and thus

$$\begin{aligned} \deg([m](c_1, \ldots, c_m)) &= m + \sum_{i=1}^{m} \deg(c_i) \\ &\leq \sum_{s=1}^{u} n_s + \sum_{s=1}^{u} \sum_{j=1}^{n_s} \deg(d_{sj}) \\ &= \sum_{s=1}^{u} \deg([n_s](d_1, \ldots, d_{n_s})). \end{aligned}$$

If $u = 1$ and if equality of degrees holds, then we must have $m = n$, whence δ_1 is the identity map, and then we must have $\deg(c_i) = \deg(d_i)$ for $i = 1, \ldots, m$, whence each f_i is an identity map. □

Remark 2.12. The Θ construction can be applied to an arbitrary multicategory $\mathcal{M}$; when the multicategory $\mathcal{M} = \mathcal{C}(*)$ for some category $\mathcal{C}$, then the construction specializes to the one we have used. Given a suitable notion of "Reedy multicategory", it seems that the above proof can be generalized to show that $\Theta\mathcal{M}$ is a Reedy multicategory whenever $\mathcal{M}$ is; we state this result in Section 5. These ideas seem to be a generalization of Angeltveit's work on enriched Reedy categories constructed from operads [1].

2.13. **The direct sub-multicategory of $\Theta\mathcal{C}$.** We give a criterion which can be useful for identifying the morphisms of $(\Theta\mathcal{C})^+$, and more generally the multimorphisms of $(\Theta\mathcal{C})^+(*)$.

Given a multimorphism $f = (f_s \colon c \to d_s)_{s=1,\ldots,u}$ in the multicategory $\mathcal{C}(*)$ associated to a category $\mathcal{C}$, let Ff denote the induced map of of $\mathcal{C}$-presheaves

$$(Ff_1, \ldots, Ff_u) \colon Fc \to Fd_1 \times \cdots \times Fd_u.$$

Proposition 2.14. *Let $\mathcal{C}$ be a multi-Reedy category, and suppose that for every f in $\mathcal{C}^+(*)$, the map Ff is a monomorphism in* $\mathrm{Psh}(\mathcal{C})$. *Then for every g in $(\Theta\mathcal{C})^+(*)$, the map Fg is a monomorphism in* $\mathrm{Psh}(\Theta\mathcal{C})$.

Proof. Let $g = (g_s)_{s=1,\ldots,u}$ be a multimorphism in $(\Theta\mathcal{C})^+(*)$, where

$$g_s = (\beta_s, g_{sj}) \colon [n](d_1, \ldots, d_n) \to [p_s](e_{s1}, \ldots, e_{sp_s}).$$

We need to show that if

$$f, f' \colon [m](c_1, \ldots, c_m) \to [n](d_1, \ldots, d_n)$$

are maps in $\Theta\mathcal{C}$ such that $g_s f = g_s f'$ for all $s = 1, \ldots, u$, then $f = f'$. Write $f = (\alpha, f_i)$ and $f' = (\alpha', f_i')$. Then $g_s f = g_s f'$ implies $\beta_s \alpha = \beta_s \alpha'$ for all s, whence $\alpha = \alpha'$ since

$$(F\beta_s) \colon F[n] \to F[p_1] \times \cdots \times F[p_u]$$

is a monomorphism in $\mathrm{Psh}(\Delta)$. Thus for each $i = 1, \ldots, m$ and $\alpha(i-1) < j \leq \alpha(i)$ we have $f_i, f_i' \colon c_i \to d_j$, which satisfy $g_{sj} f_i = g_{sj} f_i'$ for all $s = 1, \ldots, u$. By hypothesis on $\mathcal{C}$, it follows that $f_i = f_i'$. □

3. Elegant Reedy categories

In this section, we give sufficient conditions on a Reedy category to ensure that the Reedy and injective model structures agree.

3.1. **Degenerate and non-degenerate points.** Let $\mathcal{C}$ be a Reedy category, and suppose that X is an object of $\mathrm{Psh}(\mathcal{C})$.

Definition 3.2. A c-point $x \in X(c)$ is *degenerate* if there exist $\alpha \colon c \to d$ in $\mathcal{C}^-$ and $y \in X(d)$ such that

1. $(X\alpha)(y) = x$, and
2. α is not an identity map (or equivalently, $\deg(c) > \deg(d)$).

A c-point $x \in X(c)$ is *non-degenerate* if it is not degenerate.

We write $X_{\mathrm{dg}}(c), X_{\mathrm{nd}}(c) \subseteq X(c)$ for the subsets of degenerate and non-degenerate c-points of X, respectively; thus

$$X(c) = X_{\mathrm{dg}}(c) \amalg X_{\mathrm{nd}}(c).$$

If $f \colon X \to Y$ in $\mathrm{Psh}(\mathcal{C})$ is a map, then $f(X_{\mathrm{dg}}(c)) \subseteq Y_{\mathrm{dg}}(c)$, while $f^{-1}(Y_{\mathrm{nd}}(c)) \subseteq X_{\mathrm{nd}}(c)$.

Definition 3.3. A c-point $x \in X(c)$ is a *degeneracy* of $y \in X(d)$ if there exists $\alpha \colon c \to d$ in $\mathcal{C}^-$ such that $x = X(\alpha)(y)$; every point is a degeneracy of itself.

Because the slice category $(c \downarrow \mathcal{C}^-)$ is finite dimensional, every point in X is the degeneracy of at least one non-degenerate point.

For an object c in $\mathcal{C}$, a point $\alpha \in (Fc)(d)$ is non-degenerate if and only if $\alpha \colon c \to d$ is in $\mathcal{C}^+$. *Warning:* It is not the case that $\alpha \colon c \to d$ in $\mathcal{C}^+$ implies that $F\alpha \colon Fc \to Fd$ is injective.

3.4. **Elegant Reedy categories.**

Definition 3.5. A Reedy category $\mathcal{C}$ is *elegant* if the following two properties hold:

- (E1) For every presheaf Y in $\mathrm{Psh}(\mathcal{C})$ and subpresheaf $X \subseteq Y$, we have $X_{\mathrm{nd}}(c) \subseteq Y_{\mathrm{nd}}(c)$ for all objects c in $\mathcal{C}$.
- (E2) For every presheaf X in $\mathrm{Psh}(\mathcal{C})$, every object c in $\mathcal{C}$, and every c-point $x \in X(c)$, there exists a unique pair ($\sigma \colon c \to d$ in $\mathcal{C}^-$ and $y \in X_{\mathrm{nd}}(d)$) such that $(X\sigma)(y) = x$.

Condition (E1) admits the following equivalent reformulation.

(E1') For every presheaf Y in $\mathrm{Psh}(\mathcal{C})$ and subpresheaf $X \subseteq Y$, the square

$$\begin{tikzcd} X_{\mathrm{dg}}(c) \arrow[r] \arrow[d] & X(c) \arrow[d] \\ Y_{\mathrm{dg}}(c) \arrow[r] & Y(c) \end{tikzcd}$$

is a pullback for all objects c in $\mathcal{C}$.

Condition (E2) admits the following equivalent reformulation.

(E2') For every presheaf X in $\mathrm{Psh}(\mathcal{C})$ and every object c in $\mathcal{C}$, the map

$$\coprod_{d \in \mathrm{ob}(\mathcal{C})} \coprod_{x \in X_{\mathrm{nd}}(d)} \mathcal{C}^{-}(c,d) \to X(c),$$
$$(d, x, \alpha) \mapsto (X\alpha)(x)$$

is a bijection.

3.6. **Equivalence of Reedy and injective model structures.** Let $\mathcal{C}$ be a Reedy category. Given a presheaf X in $\mathrm{Psh}(\mathcal{C}, \mathcal{M})$ on $\mathcal{C}$ taking values in some cocomplete category $\mathcal{M}$, for each object c in $\mathcal{C}$ the *latching object* at c is an object L_cX of M together with a map $p_c \colon L_cX \to X(c)$, defined by

$$L_cX \overset{\mathrm{def}}{=} \mathrm{colim}_{(\alpha\colon c \to d) \in \partial(c \downarrow \mathcal{C})} X(d) \xrightarrow{(X\alpha)} X(c),$$

where $\partial(c \downarrow \mathcal{C})$ denotes the full subcategory of the slice category $(c \downarrow \mathcal{C})$ whose objects are morphisms $\alpha \colon c \to d$ which are not in $\mathcal{C}^{+}$. It is straightforward to show that the inclusion functor $\partial(c \downarrow \mathcal{C}^{-}) \to \partial(c \downarrow \mathcal{C})$ is final, so that the natural map

$$L_cX \to \mathrm{colim}_{(\alpha\colon c \to d) \in \partial(c \downarrow \mathcal{C}^{-})} X(d)$$

is an isomorphism. Note that for each object c in $\mathcal{C}$ the map p_c factors through a surjection $q_c \colon L_cX \to X_{\mathrm{dg}}(c)$.

Lemma 3.7. *Suppose $\mathcal{C}$ is an elegant Reedy category, and let X be an object of $\mathrm{Psh}(\mathcal{C})$. Then the map $q_c \colon L_cX \to X_{\mathrm{dg}}(c)$ is an isomorphism.*

Proof. Condition (E2') amounts to the observation that

$$\mathrm{colim}_{(\alpha\colon c \to d) \in \partial(c \downarrow \mathcal{C}^{-})} X(d) \to X_{\mathrm{dg}}(c)$$

is a bijection. □

Proposition 3.8. *Let $\mathcal{C}$ be a Reedy category and $\mathcal{M}$ a model category. Then the following are equivalent.*

1. *The category $\mathcal{C}$ is elegant.*
2. *For every monomorphism $f \colon X \to Y$ in $\mathrm{Psh}(\mathcal{C}, \mathcal{M})$, and every object c of $\mathcal{C}$, the induced map $g_c \colon X(c) \coprod_{L_cX} L_cY \to Y(c)$ is a monomorphism.*

Proof. Given a map $f\colon X \to Y$ in $s\mathrm{Psh}(\mathcal{C})$, and an object c of $\mathcal{C}$, we consider the following commutative diagram.

$$\begin{tikzcd}
& X_{\mathrm{nd}}(c) \amalg Y_{\mathrm{dg}}(c) \arrow[d, "u"', "\sim"] \arrow[dr, "g''_c"] & \\
X(c) \amalg_{L_cX} L_cY \arrow[r, two heads, "\bar{q}"] \arrow[rr, bend right, "g_c"'] & X(c) \amalg_{X_{\mathrm{dg}}(c)} Y_{\mathrm{dg}}(c) \arrow[r, "g'_c"] & Y(c)
\end{tikzcd}$$

The map $\bar{q}$ is a surjection induced by the maps q_c for X and Y. The isomorphism u is induced by the isomorphism $X(c) \cong X_{\mathrm{nd}}(c) \amalg X_{\mathrm{dg}}(c)$. The map q'_c can be defined using the fact that the maps q_c are isomorphisms, and then g''_c can be defined via the isomorphism u.

We first prove that (1) implies (2), i.e., if $\mathcal{C}$ is elegant and f is a monomorphism, then the map g_c is a monomorphism. Given an injective map $f\colon X \to Y$, and an object c in $\mathcal{C}$, the map $\bar{q}$ in the above diagram is an isomorphism by Lemma 3.7, and therefore we need to show that g''_c is a monomorphism. The restriction $g''_c|_{Y_{\mathrm{dg}}(c)}$ is the inclusion of $Y_{\mathrm{dg}}(c)$ in $Y(c)$, and so is injective. The restriction $g''_c|_{X_{\mathrm{nd}}(c)}$ is equal to $f|_{X_{\mathrm{nd}}(c)}$. Thus, to show that g''_c is injective it suffices to show that

(i) $f|_{X_{\mathrm{nd}}(c)}$ is injective, and
(ii) $f(X_{\mathrm{nd}}(c)) \subseteq Y_{\mathrm{nd}}(c)$.

Statement (i) follows since f is injective, and statement (ii) is (E1). Thus g''_c is injective, and thus g_c is injective.

Next we show that (2) implies (1). If $X = \varnothing$ and we consider a map $f\colon \varnothing \to Y$ and an object c in $\mathcal{C}$, then condition (2) implies that $g_c\colon L_cY \to Y(c)$ is injective, which implies that the surjection $\bar{q} = q_c\colon L_cY \to Y_{\mathrm{dg}}(c)$ is actually an isomorphism. This proves (E2').

Now suppose that $f\colon X \to Y$ is a monomorphism. Then again we use the fact that Lemma 3.7 implies

$$\bar{q}\colon X(c) \amalg_{L_cX} L_cY \to X(c) \amalg_{X_{\mathrm{dg}}(c)} Y_{\mathrm{dg}}(c)$$

is an isomorphism. Therefore, we may conclude that the map g''_c is injective, which implies that

$$f(X_{\mathrm{nd}}(c)) \cap Y_{\mathrm{nd}}(c) = g''_c(X_{\mathrm{nd}}(c)) \cap g''_c(Y_{\mathrm{nd}}(c)) = \varnothing,$$

which is to say $f(X_{\mathrm{nd}}(c)) \subseteq Y_{\mathrm{nd}}(c)$. This proves (E1). □

Proposition 3.9. *Let $\mathcal{C}$ be an elegant Reedy category, and let $\mathcal{M}$ be a model category in which the cofibrations are the monomorphisms. Then the injective and Reedy model structures on* $\mathrm{Psh}(\mathcal{C}, \mathcal{M})$ *coincide.*

Proof. Reedy cofibrations are always monomorphisms in $\mathrm{Psh}(\mathcal{C}, \mathcal{M})$ by [5, 15.7.2]. The converse statement, that monomorphisms in $\mathrm{Psh}(\mathcal{C}, \mathcal{M})$ are Reedy cofibrations, was proved for elegant Reedy categories $\mathcal{C}$ in the previous proposition. □

3.10. **Characterization of elegant Reedy categories.** The material in this section is prefigured in Gabriel-Zisman [4, §II.3].

Definition 3.11. A *strong pushout* in a category $\mathcal{C}$ is a commutative square in $\mathcal{C}$ such that its image under the Yoneda functor $F\colon \mathcal{C} \to \mathrm{Psh}(\mathcal{C})$ is a pushout square.

Note that every strong pushout is actually a pushout in $\mathcal{C}$.

Proposition 3.12. *Let $\mathcal{C}$ be a Reedy category. Then $\mathcal{C}$ is elegant if and only if the following property (SP) holds.*

(SP) Every pair of maps $\sigma_s \colon c \to d_s$, $s = 1, 2$, in $\mathcal{C}^-$, extends to a commutative square in $\mathcal{C}^-$ which is a strong pushout in $\mathcal{C}$. That is, there exist $\tau_s \colon d_s \to e$ in $\mathcal{C}^-$ such that $\tau_1\sigma_1 = \tau_2\sigma_2$ and such that

$$\begin{tikzcd}
Fc \arrow[r, "F\sigma_1"] \arrow[d, "F\sigma_2"'] & Fd_1 \arrow[d, "F\tau_1"] \\
Fd_2 \arrow[r, "F\tau_2"'] & Fe
\end{tikzcd}$$

in a pushout square in $\mathrm{Psh}(\mathcal{C})$.

We note some immediate consequences of property (SP).

(1) In a Reedy category, all isomorphisms are identity maps, and thus colimits are *unique up to identity* if they exist. Thus, the strong pushout guaranteed by property (SP) is unique up to identity.

(2) If $\sigma \colon c \to d$ is in $\mathcal{C}^-$, then $F\sigma \colon Fc \to Fd$ is a *surjective* map of presheaves. That is,

$$\operatorname{colim}(Fd \xleftarrow{F\sigma} Fc \xrightarrow{F\sigma} Fd) \xrightarrow{(F1_d, F1_d)} Fd$$

is an isomorphism. By condition (SP), there are maps $\tau_s \colon j \to k$ for $s = 1, 2$ such that $\tau_1\sigma = \tau_2\sigma$ fitting into a strong pushout square. Then there is a unique $\gamma \colon e \to d$ in $\mathcal{C}$ making the diagram

$$\begin{tikzcd}
Fc \arrow[r, "F\sigma"] \arrow[d, "F\sigma"'] & Fd \arrow[d, "F\tau_1"'] \arrow[rdd, "F1_d"] & \\
Fd \arrow[r, "F\tau_2"] \arrow[rrd, "F1_d"'] & Fe \arrow[rd, dotted, "F\gamma"] & \\
& & Fd
\end{tikzcd}$$

commute. Since $\gamma\tau_s = 1_d$ and $\tau_s \in \mathcal{C}^-$ for $s = 1, 2$, we must have that γ is an identity map, since $\mathcal{C}$ is a Reedy category.

(3) The preceding remark implies that each $\sigma \colon c \to d$ in $\mathcal{C}^-$ is a *split epimorphism.* That is, σ is a map such that there exists $\delta \colon d \to c$ in $\mathcal{C}$ such that $\sigma\delta = 1_d$. Furthermore, a morphism $\alpha \colon c \to d$ in $\mathcal{C}$ is in $\mathcal{C}^-$ if and only if $F\alpha$ is surjective; to prove the if part, note that any split epimorphism in $\mathcal{C}$ is necessarily in $\mathcal{C}^-$.

(4) The slice category $(c \downarrow \mathcal{C}^-)$ is cocomplete. Since all morphisms are epimorphisms, $(c \downarrow \mathcal{C}^-)$ is a poset. It has an initial object $1_c \colon c \to c$, and has finite coproducts by property (SP), and so has a finite colimits. Since $(c \downarrow \mathcal{C}^-)$ has finite dimensional nerve, it trivially has all filtered colimits.

Proof of Proposition 3.12. Suppose $\mathcal{C}$ is a Reedy category which satisfies property (SP). To prove (E1), let $X \subseteq Y$ be presheaves, let c be an object of $\mathcal{C}$, suppose $x \in X_{\mathrm{nd}}(c)$, and suppose that $x \notin Y_{\mathrm{nd}}(c)$. Then there exists $y \in Y_{\mathrm{nd}}(d)$ and a non-identity map $\alpha\colon c \to d$ in $\mathcal{C}^-$ such that $Y(\alpha)(y) = x$. That is, the square diagram

$$\begin{tikzcd}
Fc \arrow[r, "\bar{x}"] \arrow[d, "F\alpha"'] & X \arrow[d, hook] \\
Fd \arrow[r, "\bar{y}"'] \arrow[ur, dotted] & Y
\end{tikzcd}$$

commutes. But, we have observed that property (SP) implies that $F\alpha$ is surjective, and thus a dotted arrow exists making the triangles commute. This fact implies that $x \in X(c)$ is degenerate, contradicting the hypotheses.

To prove (E2), suppose that $x \in X(c)$, and suppose that we are given $\sigma_s\colon c \to d_s$ in $\mathcal{C}^-$ and $y_s \in X_{\mathrm{nd}}(d_s)$ for $s = 1, 2$, such that $(X\sigma_s)(y_s) = x$. Then there is a unique dotted arrow $\bar{z}$ making the diagram

$$\begin{tikzcd}
 & Fd_1 \arrow[dr, "F\tau_1"'] \arrow[drr, bend left, "\bar{y}_1"] & & \\
Fc \arrow[ur, "F\sigma_1"] \arrow[dr, "F\sigma_2"'] & & Fe \arrow[r, dotted, "\bar{z}"] & X \\
 & Fd_2 \arrow[ur, "F\tau_2"] \arrow[urr, bend right, "\bar{y}_2"'] & &
\end{tikzcd}$$

commute, where the pair of maps $\tau_s\colon d_s \to e$ in $\mathcal{C}^-$ forms the strong pushout in $\mathcal{C}$ of the original pair of maps σ_s. But since y_1 and y_2 are non-degenerate, we must have that τ_1 and τ_2 are identity maps, whence $y_1 = y_2$ and $\sigma_1 = \sigma_2$.

Next we prove that if $\mathcal{C}$ is an elegant Reedy category, then property (SP) holds. First consider $\sigma\colon c \to d$ in $\mathcal{C}^-$. We claim that $F\sigma\colon Fc \to Fd$ is a surjective map of presheaves. Let $X \subseteq Fd$ be the image of $F\sigma$; we will prove that $1_d \in X(d)$ and thus $X = Fd$. Tautologically, $\alpha \in X(c)$. By property (E2), there exist $\tau\colon c \to d'$ in $\mathcal{C}^-$ and $\delta \in X_{\mathrm{nd}}(d')$ such that $\sigma = (X\tau)(\delta)$. By property (E1), it follows that $\delta \in (Fd)_{\mathrm{nd}}(d') = \mathcal{C}^+(d', d)$. In other words, we have a factorization $\sigma = \delta\tau$ with σ, τ in $\mathcal{C}^-$ and δ in $\mathcal{C}^+$, from which it follows that δ must be the identity map of d, and thus $X = Fd$.

Now suppose that $\sigma_s\colon c \to d_s$, $s = 1, 2$, is a pair of maps in $\mathcal{C}^-$. Let X denote the pushout of $F\sigma_1$ and $F\sigma_2$ in $\mathrm{Psh}(\mathcal{C})$, with maps $\bar{y}_s\colon Fd_s \to X$ such that $\bar{y}_1(F\sigma_1) = \bar{y}_2(F\sigma_2)$. We write $y_s \in X(d_s)$ for the point corresponding to the map $\bar{y}_s$. By property (E2), there exist $\tau_s\colon d_s \to e_s$ in $\mathcal{C}^-$ and $z_s \in X_{\mathrm{nd}}(e_s)$ for $s = 1, 2$ such that $(X\tau_s)(z_s) = y_s$. Since $(X\tau_1\sigma_1)(z_1) = (X\tau_2\sigma_2)(z_2)$, the uniqueness statement of (E2) implies that $e_1 = e_2$, $z_1 = z_2$, and $\tau_1\sigma_1 = \tau_2\sigma_2$. Write $z = z_1$ and $e = e_1$, and consider the commutative diagram

$$\begin{tikzcd}
 & Fd_1 \arrow[dr, "\bar{y}_1"'] \arrow[drr, "F\tau_1"] \arrow[drrr, bend left, "\bar{y}_1"] & & & \\
Fc \arrow[ur, "F\sigma_1"] \arrow[dr, "F\sigma_2"'] & & X \arrow[r, dotted, "f"] & Fe \arrow[r, "\bar{z}"] & X \\
 & Fd_2 \arrow[ur, "\bar{y}_2"] \arrow[urr, "F\tau_2"'] \arrow[urrr, bend right, "\bar{y}_2"'] & & &
\end{tikzcd}$$

The arrow f exists because X is a pushout, and we have $\bar{z}f = 1_X$. Thus f is injective. On the other hand, we have proved that $F\tau_1$ is surjective, and thus f is surjective, and so is an isomorphism. Therefore, $(\tau_s \colon d_s \to e)_{s=1,2}$ is a strong pushout of the original pair (σ_s). ☐

4. The Eilenberg-Zilber Lemma

Let $\mathcal{C}$ be a Reedy category. Given a map $\alpha \colon c \to d$ in $\mathcal{C}$, let $\Gamma(\alpha)$ denote the set of *sections* of α; that is,

$$\Gamma(\alpha) = \{\, \beta \colon d \to c \text{ in } \mathcal{C} \mid \alpha\beta = 1_d \,\}.$$

Note that if $\sigma \colon c \to d$ is a map in $\mathcal{C}^-$, then $\Gamma(\sigma) \subseteq \mathcal{C}^+(d, c)$.

Definition 4.1. A Reedy category $\mathcal{C}$ is an *EZ-Reedy category* if the following two conditions hold.

(EZ1) For all $\sigma \colon c \to d$ in $\mathcal{C}^-$, the set $\Gamma(\sigma)$ is non-empty.
(EZ2) For all pairs of maps $\sigma, \sigma' \colon c \to d$ in $\mathcal{C}^-$, if $\Gamma(\sigma) = \Gamma(\sigma')$, then $\sigma = \sigma'$.

Note that if $\mathcal{C}$ is EZ-Reedy, then every $\sigma \colon c \to d$ in $\mathcal{C}^-$ is a split epimorphism, and therefore $F\sigma \colon Fc \to Fd$ is a surjection in $\mathrm{Psh}(\mathcal{C})$.

The following argument is due to Eilenberg and Zilber; it is proved in [4, §II.3].

Proposition 4.2. *If $\mathcal{C}$ is an EZ-Reedy category, then $\mathcal{C}$ is elegant.*

Proof. Suppose that $\mathcal{C}$ is an EZ-Reedy category. Let $f \colon X \to Y$ be a monomorphism in $\mathrm{Psh}(\mathcal{C})$, and suppose that $x \in X(c)$ and $f(x) \in Y_{\mathrm{dg}}(c)$. Thus, there exist $\sigma \colon c \to d$ in $\mathcal{C}^-$ and $y \in Y(d)$ such that $(Y\sigma)(y) = f(x)$ and σ is not an identity map. Since $\Gamma(\sigma)$ is non-empty by (EZ1), σ is a split epimorphism, and thus $F\sigma$ is a surjection in $\mathrm{Psh}(\mathcal{C})$. Therefore, a dotted arrow exists in the diagram

$$\begin{tikzcd}
Fc \arrow[r, "\bar{x}"] \arrow[d, "F\sigma"'] & X \arrow[d, "f"] \\
Fd \arrow[r, "\bar{y}"'] \arrow[ur, dotted] & Y
\end{tikzcd}$$

thus showing that $x \in X(c)$ is also degenerate. This proves property (E1).

Now suppose $x \in X(c)$, and that there are $\sigma_s \colon c \to d_s$ in $\mathcal{C}^-$ and $y_s \in X_{\mathrm{nd}}(d_s)$ such that $(X\sigma_s)(y_s) = x$, for $s = 1, 2$. For any choices of $\delta_s \in \Gamma(\sigma_s)$, we have a diagram

$$\begin{tikzcd}
 & Fd_1 \arrow[dl, bend right, "F\delta_1"'] \arrow[dr, "\bar{y}_1"] & \\
Fc \arrow[ur, "F\sigma_1"'] \arrow[rr, "\bar{x}"] \arrow[dr, "F\sigma_2"] & & X \\
 & Fd_2 \arrow[ul, bend left, "F\delta_2"] \arrow[ur, "\bar{y}_2"'] &
\end{tikzcd}$$

in which $\bar{y}_1 = \bar{y}_1(F\sigma_1)(F\delta_1) = \bar{x}(F\delta_1) = \bar{y}_2(F(\sigma_2\delta_1))$ and $\bar{y}_2 = \bar{y}_2(F\sigma_2)(F\delta_2) = \bar{x}(F\delta_2) = \bar{y}_1(F(\sigma_1\delta_2))$. Since $\bar{y}_1$ and $\bar{y}_2$ are non-degenerate points with a common degeneracy, it follows that $d_1 = d_2$, $\bar{y}_1 = \bar{y}_2$, and $\sigma_2\delta_1 = 1 = \sigma_1\delta_2$. Since δ_1 and δ_2 were arbitrary choices of sections, we see that $\Gamma(\sigma_1) = \Gamma(\sigma_2)$, and thus $\sigma_1 = \sigma_2$. Thus we have proved property (E2). □

4.3. **The category Θ_k is EZ-Reedy, and so is elegant.** If $\mathcal{C}$ is an EZ-Reedy category with a terminal object, then for any two objects c, d in $\mathcal{C}$ the set of maps $\mathcal{C}(c, d)$ is non-empty. This fact holds because any map $\beta\colon d \to 1$ to the terminal object must be in $\mathcal{C}^-$, and therefore must have a section $\delta\colon 1 \to d$. Thus $\beta\alpha \in \mathcal{C}(c, d)$, where $\alpha\colon c \to 1$.

Proposition 4.4. *Let $\mathcal{C}$ be a multi-Reedy category which has a terminal object. If $\mathcal{C}$ is an EZ-Reedy category, then so is $\Theta\mathcal{C}$.*

Proof. Fix a morphism

$$f = (\sigma, f_i)\colon [m](c_1, \dots, c_m) \to [n](d_1, \dots, d_n)$$

in $(\Theta\mathcal{C})^-$. We first determine the structure of the set of sections $\Gamma(f)$. Observe that the functor $\Theta\mathcal{C} \to \Delta$ induces a natural map $\varphi_f\colon \Gamma(f) \to \Gamma(\sigma)$. For each $\delta \in \Gamma(\sigma)$, we write $\Gamma_\delta(f)$ for the fiber of φ over δ. It is straightforward to check that $\Gamma_\delta(f)$ consists of all maps of the form $g = (\delta, g_j)$, where each $g_j = (g_{ji}\colon d_j \to c_i)_{\delta(j-1)<i\leq\delta(j)}$ is a multimorphism in $\mathcal{C}$, with the following property: for i such that $\sigma(i-1) < j = \sigma(i)$, we have $g_{ji} \in \Gamma(f_i)$.

Thus, $\Gamma_\delta(f)$ is in bijective correspondence with a subset of $\prod_{j=1}^{n} \prod_{i=\delta(j-1)+1}^{\delta(j)} C(d_j, c_i)$, namely the set $G_\delta(f) = \prod_{j=1}^{n} \prod_{i=\delta(j-1)+1}^{\delta(j)} G_{ij}(f)$, where

$$G_{ij}(f) = \begin{cases} \Gamma(f_i) & \text{if } \sigma(i-1) < j = \sigma(i), \\ \mathcal{C}(d_j, c_i) & \text{otherwise.} \end{cases}$$

Because $\mathcal{C}$ has a terminal object, every set $\mathcal{C}(d_j, c_i)$ is non-empty, and since $\Gamma(f_i)$ is non-empty by hypothesis, we have that $\Gamma_\delta(f)$ is non-empty for each $\delta \in \Gamma(\sigma)$. Thus φ_f is surjective, and since $\Gamma(\sigma)$ is non-empty, this proves (EZ1).

Now suppose $f = (\sigma, f_i)$ and $f' = (\sigma', f_i')$ are two maps $[m](c_1, \dots, c_m) \to [n](d_1, \dots, d_n)$ in $(\Theta\mathcal{C})^-$, and that $\Gamma(f) = \Gamma(f')$. Since $\phi_f\colon \Gamma(f) \to \Gamma(\sigma)$ and $\phi_{f'}\colon \Gamma(f') \to \Gamma(\sigma')$ are surjective, we must have $\Gamma(\sigma) = \Gamma(\sigma')$, and thus $\sigma = \sigma'$. Thus, for each i and j such that $\sigma(i-1) < j = \sigma(i)$, we have maps $f_{ij}, f_{ij}'\colon c_i \to d_j$. For each $\delta \in \Gamma(\sigma)$ we therefore have $\Gamma_\delta(f) = \Gamma_\delta(f')$, which must therefore both correspond to the same subset of

$$\prod_{j=1}^{n} \prod_{\delta(j-1)+1}^{\delta(j)} \mathcal{C}(d_j, c_i).$$

Therefore $G_{ij}(f) = G_{ij}(f')$ for all $\delta(j-1) < i \leq \delta(j)$. In particular, for every $i = 1, \dots, n$ such that $\sigma(i-1) < \sigma(i)$, we have that $\Gamma(f_i) = G_{i\sigma(i)}(f) = G_{i\sigma(i)}(f') = \Gamma(f_i')$, and hence $f_i = f_i'$ since $\mathcal{C}$ is EZ-Reedy. □

Corollary 4.5. *For all $k \geq 0$, Θ_k an elegant Reedy category, with direct and inverse subcategories as described in the introduction.*

Proof. Proposition 2.11 allows us to put a Reedy model category structure on each Θ_k, and we have shown that with this structure, every $\alpha\colon \theta \to \theta'$ in Θ_k^+ induces a monomorphism $F\alpha\colon F\theta \to F\theta'$ of presheaves.

Induction on k, together with the fact that each Θ_k has a terminal object, shows that each Θ_k with the multi-Reedy structure is an EZ-Reedy category. This shows that Θ_k is elegant, and also shows that every $\alpha\colon \theta \to \theta'$ in Θ_k^- induces an epimorphism $F\alpha\colon F\theta \to F\theta'$ of presheaves.

Since any map of presheaves factors uniquely, up to isomorphism, into an epimorphism followed by a monomorphism, this shows that Θ_k^- and Θ_k^+ must be exactly the classes of maps described in the introduction. □

5. Reedy multicategories

With the definition of multi-Reedy category, which is a multicategory arising from a Reedy category, one might ask whether the notion of Reedy category can be extended to that of a Reedy multicategory. In this section, we propose a definition and give examples.

Definition 5.1. A *Reedy multicategory* is a symmetric multicategory $\mathcal{A}$ equipped with the following structure: wide submulticategories $\mathcal{A}^-$ and $\mathcal{A}^+$, where $\mathcal{A}^-$ only has multimorphisms with valence at most 1, and $\mathcal{A}^+$ only has multimorphisms with valence at least 1, together with a function $\deg\colon \mathrm{ob}(\mathcal{A}) \to \mathbb{N}$ such that the following properties hold.

1. For all $f\colon a \to b_1, \dots, b_m$ in $\mathcal{A}$ with $m \geq 1$, there exists a unique factorization in $\mathcal{A}$ of the form $f = f^+ f^-$, where f^- is in $\mathcal{A}^-$ and f^+ is in $\mathcal{A}^+$.
2. For all $f\colon a \to b$ in $\mathcal{A}^-$, we have $\deg(a) \geq \deg(b)$, with equality if and only if f is an identity map. For all $f\colon a \to b_1, \dots, b_m$ in $\mathcal{A}^+$, we have $\deg(a) \leq \sum_{i=1}^m \deg(b_i)$; when $m = 1$, equality holds if and only if f is an identity map.

Note that the underlying category of $\mathcal{A}$ is a Reedy category.

Example 5.2. Let $\mathcal{A}$ be a symmetric multicategory with one object, i.e., an operad. Write $\mathcal{A}_n$ for $\mathcal{A}(a; a, \dots, a)$, where a is the unique object. Suppose that $\mathcal{A}_0 = \mathcal{A}_1 = *$. Then $\mathcal{A}$ can be given the structure of a Reedy multicategory, with $\mathcal{A}_k^- = \mathcal{A}_k$ for $k \leq 1$ and $\mathcal{A}_k^+ = \mathcal{A}_k$ for $k \geq 1$, and with $\deg(a) = 0$.

Given a symmetric multicategory $\mathcal{A}$, we define a category $\Theta\mathcal{A}$ as follows. The objects of $\Theta\mathcal{A}$ are

$$[m](a_1, \dots, a_m), \qquad m \geq 0, \quad a_1, \dots, a_m \in \mathrm{ob}(\mathcal{A}).$$

The morphisms

$$f\colon [m](a_1, \dots, a_m) \to [n](b_1, \dots, b_n)$$

are given by data $(\alpha, \{f_i\})$, where $\alpha\colon [m] \to [n]$ in Δ, and for each $i = 1, \dots, m$, $f_i\colon a_i \to b_{\delta(i-1)+1}, \dots, b_{\delta(i)}$ in $\mathcal{A}$.

The category $\Theta\mathcal{A}$ can be extended to a multicategory. A multimorphism

$$f\colon [m](a_1, \dots, a_m) \to [n_1](b_{11}, \dots, b_{1n_1}), \dots, [n_u](b_{u1}, \dots, b_{un_u})$$

is given by $(\alpha, \{f_i\})$ consisting of a multimorphism $\alpha\colon [m] \to [n_1], \dots, [n_u]$ in Δ, i.e., a sequence of morphisms $\alpha_s\colon [m] \to [n_s]$ in Δ, and for each $i = 1, \dots, m$ a multimorphism

$$f_i\colon a_i \to b_{1,\alpha_1(i-1)+1}, \dots, b_{1,\alpha_1(i)}, \dots, b_{u,\alpha_u(i-1)+1}, \dots, b_{u,\alpha_u(i)};$$

i.e., a multimorphism with target $(b_{sj})_{1 \leq s \leq u,\ \delta_s(i-1) < j \leq \delta_s(i)}$.

As remarked above, the same sort of argument used to prove Proposition 2.11 can be used to establish the following result.

Proposition 5.3. *$\Theta\mathcal{A}$ is a Reedy multicategory.*

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Department of Mathematics, University of California, Riverside, CA 92521
E-mail address: `bergnerj@member.ams.org`

Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL
E-mail address: `rezk@math.uiuc.edu`