Math 9C Homework 6

Commonly Asked Questions

1. Solution to the homework problem of the day: Find s_3 for the series

$$\sum_{n=1}^{\infty} \frac{1}{n^4}$$

and find upper and lower bounds for R_3.

We get that

$$s_3 = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} = 1 + \frac{1}{16} + \frac{1}{81} = \frac{1393}{1296}$$

Since this series is a p-series with $p = 4$, we know that it satisfies the conditions for the integral test, and converges. Thus, we can use the integral test remainder estimates to find upper and lower bounds for $R_3 = s - s_3$:

$$\int_4^{\infty} \frac{dx}{x^4} \leq R_3 \leq \int_3^{\infty} \frac{dx}{x^4}.$$

Evaluating the integral on the left-hand side, we get

$$\int_4^{\infty} \frac{dx}{x^4} = \lim_{tto\infty} \int_4^t \frac{x^{-4}}{=} \lim_{t\to\infty} -\frac{1}{x} x^{-3}|_4^t = \lim_{t\to\infty} -\frac{1}{3t^3} + \frac{1}{3(4^3)} = \frac{1}{3(4^3)} = \frac{1}{192}.$$

Similarly, integrating from 3 instead of from 4, we get that the right-hand side is equal to

$$\frac{1}{3(3^3)} = \frac{1}{81}.$$

Thus, we conclude that

$$\frac{1}{192} \leq R_3 \leq \frac{1}{81}.$$

In other words, the difference between s_3 (as calculated above) and the actual sum is somewhere between $\frac{1}{192}$ and $\frac{1}{81}$.

2. Why does this estimation theorem not apply to all series?

   This estimation theorem really makes use of two facts: first, that the terms are alternating, and second, that they are decreasing in size. The alternating part tells you that if you add something positive, then you'll be subtracting something next, and the decreasing part tells you that you won't subtract off any more than you just added. Then, after subtracting, you add something more, but not any more than you just removed. It is this behavior that allows one to make the conclusion for this test.

3. When is an alternating series not convergent?

   As with our other tests, in general the alternating series test only tells us when an alternating series is convergent, and when this test fails, we can't conclude anything. However, of the conditions that we are checking in this test, one of them can help us to determine divergence, and that is the requirement that the limit of the sequence $\{b_n\}$ go to zero. Recall that for a *series* to converge, the limit of the underlying *sequence* must go to zero. If the limit of $\{b_n\}$ is not zero, we don't know immediately that the limit of $\{a_n\}$ is zero, but it is often easy to check that it also has a nonzero limit. From there, we know the series cannot converge.