Math 9C Homework 3

Commonly Asked Questions

1. What is a harmonic series? How does it work? Why it is divergent? How is it different from a regular series? Is there an easier way to sum up harmonic series?

   The "harmonic series" is just a name for the series

   $$\sum_{n=1}^{\infty} \frac{1}{n}.$$

   It is an important example, because, although the terms $\frac{1}{n}$ are getting very small as n gets large, the series diverges. Thus, just knowing that the limit of the *sequence* goes to 0 is not enough to know that the *series* converges.

   The idea behind the proof that the book gives is that you can keep finding strings of terms which, when added together, are greater than $\frac{1}{2}$. Although these strings get longer and longer, there will be infinitely many of them, and thus we will be adding $\frac{1}{2}$ infinitely many times, and hence there is no way we can possibly converge to a fixed number.

   There isn't really an easier way to sum it up, and mainly the explanation is there so that you believe it, because we will frequently use the fact that this series diverges. If you understand why it diverges, it will help you to remember it.

2. How is it that one can sum up an infinite sequence?

   That is really the trick in understanding series! It is difficult to understand intuitively at first. It seems that it would be impossible to sum up infinitely many numbers, and for that reason we have to give a definition of what it means. We define an infinite sum to exist if the sequence of partial sums converges as a sequence. In other words, we take what we do understand, how to add finitely many numbers together, and then ask what happens if we keep adding more and more of the terms together. The tools we have already learned for understanding sequences allows us to make sense of this kind of limit.

3. What is the difference between a convergent sequence and a convergent series?

   To show that a sequence $\{a_n\}$ converges, we ask if the limit

   $$\lim_{n\to\infty} a_n$$

   exists. In other words, what is happening to the list of numbers $a_1, a_2, \ldots, a_n, \ldots$ as n gets large?

   When we look at series, instead of just considering the list, we ask what happens when we add those terms together. Now, we are asking, not if the sequence $\{a_n\}$ converges, but if the sequence of partial sums $\{s_n\}$ converges, where $s_n = a_1 + a_2 + \cdots + a_n$.

4. Do limit properties apply to series as well?

   Theorem 8 tells us that we can add convergent series, and multiply them by constants, in the way that you would expect. Multiplying series is more complicated, and we'll discuss it later in the chapter.

5. Are there any real-life applications of series?

   Yes. One important example that uses geometric series is that of annuities. If you save money, it earns interest, so if you want a particular amount in a few years, then you can save less than that amount now to have it later. (You probably did some problems like this when you learned about exponential functions.) What if you want to invest enough money now so that you can withdraw a certain amount each year, and have it last as long as you might possibly live? To find this amount, you are basically finding the sum of a geometric series. The same idea applies to donating money, say for scholarships, so that some money can be awarded each year but there will always be some left to earn interest and hence have money for an award the next year also.

6. Homework problem: Is the sequence given by

   $$a_n = \frac{1}{2n+3}$$

   increasing, decreasing, or neither?

   Many of you wrote down the first few terms and concluded that this sequence is decreasing. While that helps you make a good guess, you really need to show that the sequence is *always* decreasing, i.e., that $a_{n+1} < a_n$ for all n.

To do this, look at $a_n = \frac{1}{2n+3}$ and

$$a_{n+1} = \frac{1}{2(n+1)+3} = \frac{1}{2n+5}.$$

Since $2n+3 < 2n+5$, when we take reciprocals we get

$$\frac{1}{2n+5} < \frac{1}{2n+3},$$

so $a_{n+1} < a_n$.

Since the sequence is decreasing, we know that it is bounded above by the first term, $a_1 = \frac{1}{5}$. It is also bounded below by 0, since the terms will never be negative.