Math 9C Homework 1

Commonly Asked Questions

Since this reading assignment was to review the previous one, there weren't many new questions, but I want to discuss the question from class and the reading question.

1. Determine whether the sequence

$$a_n = e^{\frac{1}{n}}$$

converges or diverges, and if it converges, find the limit.

Most people got the correct answer for this problem, but I wasn't convinced that most of you understood why it was a correct answer. We know that $\lim_{n\to\infty} \frac{1}{n} = 0$, but we need to understand why we can move the limit inside the function, to take the limit of just the exponent and get the correct answer. In other words, we need to understand why

$$\lim_{n\to\infty} e^{\frac{1}{n}} = e^{\lim_{n\to\infty} \frac{1}{n}}.$$

But, we can do this using Theorem 7, since the function $f(x) = e^x$ is continuous, and we are interested in knowing $\lim_{n\to\infty} f(\frac{1}{n})$, which is just $f(\lim_{n\to\infty} \frac{1}{n}) = f(0) = e^0 = 1$.

Now, on a quiz or exam you wouldn't necessarily have to write out that whole paragraph, but you should write something like the following: Since e^x is continuous,

$$\lim_{n\to\infty} e^{\frac{1}{n}} = e^{\lim_{n\to\infty} \frac{1}{n}} = e^0 = 1.$$

A common problem was to confuse when you have to take the limit. Some people wrote things like $\lim_{n\to\infty} e^0$, but if you've gotten the exponent 0, then you've taken the limit already and shouldn't still have a limit in front. Another difficulty was writing a step that looked like $e^{\frac{1}{\infty}}$. Yes, we are thinking of what happens as n goes to ∞, but it is incorrect to write $\frac{1}{\infty}$ as if it were a number. You need to take a limit as n goes to ∞, and notice that you end up with 0. You can run into problems otherwise.

2. List methods for determining whether a sequence converges or diverges.

   Here, lots of people gave the definition of convergence or divergence of a limit. That is a good start, as you need to know what it means. But, just having a definition isn't always helpful in solving concrete examples. If you asked me how to solve a specific problem, and I told you that you need to read the definition again, you probably wouldn't feel like I'd given you a very good answer! You'd want me to give you a suggestion on a specific method so that you could find that limit. So, what I was really after here was a list of specific methods that we've used to find limits of sequences. Here's what I came up with:

   - Looking at a function $f(x)$ such that $f(n) = a_n$ for positive integers n, and using Theorem 3, which says that $\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$. Then, you might end up using tools such as l'Hopital's rule that you have developed for finding limits of functions.
   - If the terms of your sequence can be broken down into sums or products or powers of more basic sequences that are already familiar, then you can use limit laws.
   - The squeeze theorem, especially when you have terms defined using processes like factorials that don't extend to functions $f(x)$ for real numbers.
   - Taking the absolute value of the terms of your sequence, using Theorem 6.
   - If the terms are given by fractions, dividing by the highest power of n in the numerator and denominator. This process is an alternative to passing to the function and using l'Hopital's rule for some kinds of functions.
   - If the terms of our sequence are given by $f(a_n)$ for some continuous function $f(x)$ and some familiar sequence with terms a_n, we can use Theorem 7.
   - Monotonic sequence theorem.

     Notice that writing out several terms of the sequence and graphing the sequence are NOT ways to find the limit. They might help you to visualize the sequence, or help you to make a good guess, but you need to use a more precise method to find the actual limit and know that it is correct.