Math 9C Homework 16
Commonly Asked Questions

1. Solution to the in-class problem: Find the arc length function for the curve given by
$$y = \ln(1 - x^2)$$
starting at $x = 0$.

   Since we are going to find an arc length *function*, we begin by changing the variable x to t, so that we will end up with a function of x in the end. Thus, we get
$$y = \ln(1 - t^2).$$
   Taking the derivative, we get
$$\frac{dy}{dt} = \frac{-2t}{1 - t^2},$$
   and squaring it we get
$$\left(\frac{dy}{dt}\right)^2 = \frac{4t^2}{(1 - t^2)^2}.$$
   Adding one, find a common denominator and simplify as follows:
$$\begin{aligned} 1 + \left(\frac{dy}{dt}\right)^2 &= 1 + \frac{4t^2}{(1 - t^2)^2} \\ &= \frac{(1 - t^2)^2 + 4t^2}{(1 - t^2)^2} \\ &= \frac{1 - 2t^2 + t^4 + 4t^2}{(1 - t^2)^2} \\ &= \frac{1 + 2t^2 + t^4}{(1 - t^2)^2} \\ &= \frac{(1 + t^2)^2}{(1 - t^2)^2}. \end{aligned}$$
   Now, to get the arc length function starting at $x = 0$, we take
$$s(x) = \int_0^x \sqrt{\frac{(1 + t^2)^2}{(1 - t^2)^2}}\,dt = \int_0^x \frac{1 + t^2}{1 - t^2}\,dt.$$

To take this integral, we need to change the form of this fraction. First, we can split up the numerator to get

$$\frac{1+t^2}{1-t^2} = \frac{t^2}{1-t^2} + \frac{1}{1-t^2}.$$

The first part needs to be divided out, using long division, and the result is

$$\frac{t^2}{1-t^2} = \frac{1}{1-t^2} - 1.$$

Thus, our integral can now be written as

$$s(x) = \int_0^x \left(\frac{1}{1-t^2} + \frac{1}{1-t^2} - 1 \right) dt = \int_0^x \left(\frac{2}{1-t^2} - 1 \right) dt = 2\int_0^x \frac{1}{1-t^2} dt - \int_0^x dt.$$

To compute the first integral, check the formulas in the back of the book to find

$$\int \frac{du}{a^2-u^2} = \frac{1}{2a} \ln \left| \frac{u+a}{u-a} \right| + c.$$

Here, we have $a = 1$ and $u = t$. Thus, computing both pieces we get

$$s(x) = \left(\ln \left| \frac{t+1}{t-1} \right| - t \right) |_0^x = \ln \left| \frac{x+1}{x-1} \right| - x - \left(\ln \left| \frac{1}{-1} \right| - 0 \right) = \ln \left| \frac{x+1}{x-1} \right| - x.$$