MATH 875 ALGEBRAIC TOPOLOGY MIDTERM EXAM

OCTOBER 26, 2006

This midterm is due in class on Tuesday, October 31. You are permitted to use your class notes, homework, and Hatcher's book. Please do not use other references or discuss these problems with anyone else. (You may email me over the weekend, or see me on Monday if you have questions.) Good luck!

(1) Let X be a CW-complex, and let CX be its cone. Show that the pair (CX, X) has the homotopy extension property with respect to any space.

(2) Let X be the union of two simply connected subspaces whose intersection is nonempty and path connected. Prove that X is simply connected.

(3) Let G be a path connected and locally path connected topological space with a given continuous multiplication map $\mu\colon G \times G \to G$ with unit e such that $\mu(e, g) = \mu(g, e) = g$ for any $g \in G$. Let $p\colon \widetilde{G} \to G$ be a covering space with $\widetilde{G}$ also path connected and locally path connected, and $\widetilde{e} \in \widetilde{G}$ such that $p(\widetilde{e}) = e$. Prove that there exists a continuous multiplication $\widetilde{\mu}\colon \widetilde{G} \times \widetilde{G} \to \widetilde{G}$ with unit element $\widetilde{e}$ (i.e., $\widetilde{\mu}(\widetilde{e}, \widetilde{g}) = \widetilde{\mu}(\widetilde{g}, \widetilde{e}) = \widetilde{g}$ for any $\widetilde{g} \in \widetilde{G}$) such that for any $(\widetilde{g}, \widetilde{h}) \in \widetilde{G} \times \widetilde{G}$, we have that $\mu(p\widetilde{g}, p\widetilde{h}) = p\widetilde{\mu}(\widetilde{g}, \widetilde{h})$. [Hint: If $p_X : \widetilde{X} \to X$ and $p_Y : \widetilde{Y} \to Y$ are covering spaces, then $p_X \times p_Y : \widetilde{X} \times \widetilde{Y} \to X \times Y$ is a covering space.]

(4) We have used the fact that any closed 2-manifold can be written in the form $X \# T^2 \# \cdots \# T^2$, where X is either S^2, K^2, or $\mathbb{R}P^2$. Use homology and the Euler characteristic to determine how to write $K^2 \# \mathbb{R}P^2$ in one of these forms. (Here you may use the simplicial or "polyhedral" homology that we used initially for calculations; you should NOT use singular homology!)

(5) Suppose that the following diagram of abelian groups and abelian group homomorphisms is commutative with exact rows and the indicated maps isomorphisms:

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & C_{q+1} & \xrightarrow{\partial} & A_q & \xrightarrow{f} & B_q & \xrightarrow{g} & C_q & \xrightarrow{\partial} & A_{q-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle\cong} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\cong} & & \downarrow & & \\
\cdots & \longrightarrow & C'_{q+1} & \xrightarrow{\partial'} & A'_q & \xrightarrow{f'} & B'_q & \xrightarrow{g'} & C'_q & \xrightarrow{\partial'} & A'_{q-1} & \longrightarrow & \cdots
\end{array}$$

Show that there is a long exact sequence of the form

$$\cdots \to A_q \to A'_q \oplus B_q \to B'_q \to A_{q-1} \to A'_{q-1} \oplus B_{q-1} \to B'_{q-1} \to \cdots .$$