An Algebraic View of Rates of Change

Yesterday we looked at how to find average rates of change and instantaneous rates of change on the graph of a function. The average rate of change between two points on the graph was the slope of the secant line connecting those two points, and the instantaneous rate of change at a point was the slope of the tangent line to that point.

We also discussed that the slope of the tangent line is the limit of the secant lines as one point remained fixed and the other point varied and got closer and closer to the one that was fixed. We observed, for example, that the slope of the tangent line to the curve $y = x^2$ at the point $(2, 4)$ appeared to be 4. Someone asked, however, if there was an easier way to find it. Today, we'll discuss how to find the slope of the tangent line algebraically.

First, let's revisit our definition of the derivative, or the slope of the tangent line at a point $x = b$:

$$f'(b) = \lim_{a \to b} \frac{f(b) - f(a)}{b - a}.$$

Here b was the x-value of the point we were keeping fixed, and a was the point that was varying, as indicated by the $a \to b$ under the limit.

First of all, given a function $f(x)$, we'd like to consider the "derivative function" $f'(x)$. What does this mean? For any value of x, $f'(x)$ is the function whose input is x and whose output is the slope of the tangent line to the curve $y = f(x)$ at the point $(x, f(x))$.

(Something to note here is that $f'(x)$ may not have the same domain as the original function $f(x)$. Recall that yesterday we talked about examples of points where the derivative did not exist, since the tangent line did not exist.)

First of all, given the graph of a function $f(x)$, we can sketch what the graph of $f'(x)$ will look like by guessing what the slopes of the tangent lines are.

Our first example that we can compute is that of a line. Recall from yesterday that the tangent line to a point on a line $y = mx + b$ is just the line itself. In other words, at any x-value, the slope of the tangent line at $(x, f(x))$ is just the slope m of the line. So, for any $f(x) = mx + b$, $f'(x) = m$.

As an example, consider the line $y = 2x - 5$. The slope of this line is 2, so the slope of the tangent line at any point is 2. Thus, $f'(x) = 2$ for any x.

If you put in the function $f(x) = x^2$, for example, you will get $f'(x) = 2x$. In general, we get that if

$$f(x) = x^n$$

for any power n, then

$$f'(x) = nx^{n-1}.$$

Notice this fact agrees with our examples of a line (using that $x^0 = 1$), and $f(x) = x^2$. So, we know that, for example, if $f(x) = x^5$, then we can compute directly that $f'(x) = 5x^4$.

We also have that the derivative function for a constant function, for example, $f(x) = 2$, is 0, since the slope of the line $f(x) = 2$ is 0.

One question to think about: What happens if you add two functions together? What does that do to the derivative function? What if you multiply by a constant function?

Exercise 1. What is the derivative function $f'(x)$ for the function

$$f(x) = 3x^2 - 6x - 2?$$

What about for

$$g(x) = 6x^6 - 6x - x^{-4}?$$

Yesterday we also noticed that the slope of the tangent line is 0 at a turning point, i.e., a relative maximum or minimum. So, algebraically, we can compute a relative maximum or minimum using the derivative function.

As an example, consider the function $f(x) = 3x^2 + 6x + 1$. This function is a parabola with a minimum point, and we want to know where this minimum is. To find it, let's first take the derivative to get $f'(x) = 6x + 6$. To find the minimum, set this function equal to zero, so $6x + 6 = 0$. Solving for x, we get $6x = -6$, or $x = -1$. Thus, the minimum point occurs when $x = -1$. The y-value we can get from plugging in $x = -1$ into the original function, so

$$f(-1) = 3(-1)^2 + 6(-1) + 1 = 3 - 6 + 1 = -2.$$

In this same example, notice that when $x > -1$, the derivative is positive. This means that the slope of the tangent line is positive, or that the function in *increasing*. Similarly, when $x < -1$, the derivative is negative, or the slope of the tangent line is negative and the function is *decreasing* there.

This is actually a general rule: the function is increasing wherever the derivative is positive and decreasing whenever the derivative is negative. This fact can be used to understand graphs of functions and to be able to sketch them.

Exercise 2. Find the maximum point of the function

$$f(x) = -2x^2 - 9x + 4.$$

Where is this function increasing, and where is it decreasing? Explain how you got your answer.