Rates of Change

Recall that the slope of a line measures a rate of change. In particular, for a line $y = mx + b$ the slope is

$$m = \frac{\Delta y}{\Delta x},$$

the change in y divided by the change in x. So, the slope is telling you how much the "outputs" of the function (the y-values) are changing *relative* to how much the "inputs" of the function (the x-values) are changing.

Can we measure rates of change for more general functions? In other words, what is the "slope" of a graph which is not a line? At first glance it seems difficult to define because it depends where you look.

One answer is to take an *average rate of change* by taking two points on the graph and finding the change in y divided by the change in x. For example consider the function $f(x) = x^2$ and its graph. Two points on this parabola are $(0, 0)$ and $(2, 4)$. The average rate of change here is

$$\frac{\Delta y}{\Delta x} = \frac{4}{2} = 2.$$

But, what if we picked two different points, say $(1, 1)$ and $(2, 4)$? What is the average rate of change between these two points? It is

$$\frac{\Delta y}{\Delta x} = \frac{3}{1} = 3.$$

So the average rates of change are different and depend on which points we choose. Thus, it is not constant for this function, unlike for a line.

For example, suppose that you drive 600 miles in 10 hours. Then your average speed is

$$\frac{600}{10} = 60$$

miles per hour. But, we've only just calculated your *average* speed. Maybe you actually drove 75 miles per hour for 4 hours, then stopped for lunch for 2 hours (i.e., drove 0 miles per hour for 2 hours), then drove 75 miles per hour for another 4 hours. So, while 60 miles per hour is your average speed, it doesn't accurately tell you at what speed you were actually travelling at any given time.

How could you tell? Let's go back to the example of $f(x) = x^2$. We could choose a point still closer to $(2, 4)$, say $\left(\frac{3}{2}, \frac{9}{4}\right)$. Then our average rate of change is

$$\frac{\Delta y}{\Delta x} = \frac{4 - \frac{9}{4}}{2 - \frac{3}{2}} = \frac{\frac{16}{4} - \frac{9}{4}}{\frac{4}{2} - \frac{3}{2}} = \frac{\frac{7}{4}}{\frac{1}{2}} = \frac{7}{4} \cdot 2 = \frac{7}{2} = 3\frac{1}{2}.$$

The idea here is that we are getting a sequence of average rates of change, where the point $(2, 4)$ is fixed but where the other point is getting progressively closer to it. If you look at a table, where you compare points getting closer to $(2, 4)$ with the slope of the line connecting that point to $(2, 4)$, you will see that the slopes are approaching 4. (It might be a good idea to check this on a graphing calculator!)

When we choose two points on a graph and connect them with a line, this line is called a *secant line*. When we take the limit (that is really what we are doing here!) of these secant lines as the points get closer and closer together, we get the *tangent line* at the point we have fixed. Notice that, at least nearby, the secant line passes through the graph at 2 points, whereas the tangent line passes through it at only 1 point. The slope of the tangent line is what we will consider the *slope of the graph* at that point, or the *instantaneous rate of change* at that point.

More formally, suppose we have a function $f(x)$ and fix an x-value b. If we choose another point a on the graph $y = f(x)$, then the average rate of change between these two points, or the slope of the secant line between them, is given by

$$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}.$$

As we let a move closer and closer to b, we can take the limit

$$\lim_{a \to b} \frac{f(b) - f(a)}{b - a}$$

which is the slope of the tangent line.

Part of the idea behind looking at tangent lines is that, if you zoom in enough, any curve looks linear. (Think of the fact that the earth looks flat to us, but if you go out into space you can see that it isn't.)

The slope of the tangent line of $y = f(x)$ at the point $x = a$ is called the *derivative of f at a*, denoted $f'(a)$. If $f'(a)$ exists, then $f(x)$ is *differentiable at $x = a$*. The process of finding derivatives is called *differentiation*.

Consider turning points of functions. What can you say about their tangent lines?

Exercise 1. Consider the function $f(x) = 2x^2 + 1$. Find the average rate of change (or slope of the secant line) between the point $(1, 3)$ and each of the points $(-2, 5)$, $(-1, 3)$, and $(0, 1)$, and then for some other point near $(1, 3)$. Make an estimate of what you think the instantaneous rate of change (or slope of the tangent line) will be.

Exercise 2. Draw the graph of a function $f(x)$ with marked points a, b, c, d, e such that

- $f(x)$ is continuous and differentiable at $x = a$.
- $f(x)$ is continuous but not differentiable at $x = b$.
- $f(x)$ is not continuous at $x = c$ but the limit exists at $x = c$.
- The limit of $f(x)$ as x approaches d does not exist.
- The limit of $f(x)$ as x approaches e exists but $f(e)$ is not defined.