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Thirteen Ways of Looking at a Topological Group

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http://www.math.ksu.edu/~jbergner/GroupTalk.pdf

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Topological Groups

Definition 1. A *topological group* is a topological space X which is also a group. In particular the map given by multiplying by a fixed $x \in X$ is continuous.

Topological groups appear all over mathematics, since they combine both algebraic and topological information. Lie groups, which have additional structure, are particularly important.

We'd like to consider mathematical structures which perhaps look different but are equivalent to topological groups.

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Classical models for topological groups

Loop spaces (Thomason):

Given a topological space X with a specified base point, its *loop space* ΩX is the space of basepoint-preserving maps $S^1 \to X$.

Nice relationship between homotopy groups:

$$\pi_i(\Omega X) \cong \pi_{i-1}(X)$$

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Loops can be composed, but this operation is not associative:

However, given any loop space, one can find a topological group with the same homotopy type.

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Group-like A_∞-spaces (Stasheff):

Topological spaces with a group structure only up to higher homotopies.

A basic way to define a group structure is element-wise. Given elements a and b, what is ab defined to be?

Can we define a group structure in a more general way?

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An Alternative Viewpoint - Algebraic Theories

Consider the category $\mathcal{G}$ with objects groups and morphisms group homomorphisms. Then consider the subcategory whose objects are the finitely generated free groups F_n for $n \geq 0$, and with all morphisms from $\mathcal{G}$.

Notice that F_n is the free product of n copies of F_1:

$$F_n \cong \underbrace{F_1 * \cdots * F_1}_{n}.$$

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What are the maps $F_1 \to F_2$? Let's assume that $F_1 =< x >$ and $F_2 =< x, y >$.

Then such a map is given by where x goes. So, we could have

$$x \mapsto x$$

$$x \mapsto y$$

$$x \mapsto x^2 y^{-3}$$

and so forth.

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Formally, let's take the opposite of this subcategory, or reverse the direction of all the arrows. Call this opposite category $\mathcal{T}_G$, the *theory of groups*.

The maps $x \mapsto x$ and $x \mapsto y$, when reversed, can be considered to be projection maps. The other maps are various "multiplication" maps.

Now, when we take the opposite category, the free products (or coproducts) become products. Thus, in $\mathcal{T}_G$, we have an isomorphism $F_n \cong (F_1)^n$ induced by projections.

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We take functors $\mathcal{T}_G \to \mathcal{S}paces$ which preserve these products, which we call $\mathcal{T}_G$*-algebras*.

Take a product-preserving functor $A : \mathcal{T}_G \to \mathcal{S}paces$. The image of F_1 will be some space, say Y.

Then $A(F_2)$ is required to be $Y \times Y$, and similarly, $A(F_n) \cong Y^n$.

The maps between these spaces induce a group structure on $A(F_1) = Y$.

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Theorem 2. *(Lawvere, Quillen) There is an equivalence of categories*

$$\{\mathcal{T}_G - algebras\} \leftrightarrow \{topological\ groups\}.$$

Thus, $\mathcal{T}_G$-algebras are another way of looking at topological groups.

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We can instead take functors $X : \mathcal{T}_G \to \mathcal{S}paces$ which preserve products only up to homotopy. In other words, we have a map $X(T_n) \to X(T_1)^n$ which is a weak homotopy equivalence, or map inducing isomorphisms on all the homotopy groups.

We call such a functor a *homotopy $\mathcal{T}_G$-algebra.*

Notice that we can regard a (strict) $\mathcal{T}_G$-algebra as a homotopy $\mathcal{T}_G$-algebra.

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Theorem 3. *(Badzioch) Given a homotopy $\mathcal{T}_G$-algebra X, there exists a strict $\mathcal{T}_G$-algebra LX and a map $X \to LX$, which can be considered an equivalence of homotopy $\mathcal{T}_G$-algebras.*

In other words, X can be "rigidified" to a strict $\mathcal{T}_G$-algebra in a canonical way.

Furthermore, when we work with homotopy algebras rather than strict ones, we aren't losing too much information.

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Replacing spaces with simplicial sets

Simplicial sets are generalizations of simplicial complexes and are considered combinatorial models for spaces.

Formally, a simplicial set is a functor $\mathbf{\Delta}^{op} \to \mathcal{S}ets$, where $\mathbf{\Delta}^{op}$ is the category of finite ordered sets $[n] = \{0 \to 1 \to \cdots \to n\}$ and order-preserving maps between them.

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All the constructions we have done so far can be applied to simplicial sets as well as to spaces. Thus, we can define simplicial groups to be simplicial sets with a group structure.

We can also consider $\mathcal{T}_G$-algebras as product-preserving functors from $\mathcal{T}_G$ to $\mathcal{SS}ets$, the category of simplicial sets.

In this case, we still have the same relationship between strict and homotopy algebras.

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Finding a Simpler Model

The category $\mathcal{T}_G$ has a lot of morphisms, since there are a lot of homomorphisms between free groups.

Is there a diagram like $\mathcal{T}_G$, but with fewer morphisms, which also encodes the group structure?

Let's try $\mathbf{\Delta}^{op}$. Given a functor $X : \mathbf{\Delta}^{op} \to \mathcal{S}paces$, we will write X_n for $X[n]$.

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Unlike for $\mathcal{T}_G$, we do not have that $[n]$ is isomorphic to $[1]^n$, but we can consider functors for which X_n has a product condition.

Definition 4. A *Segal monoid* is a functor $X : \mathbf{\Delta}^{op} \to \mathcal{S}paces$ such that $X_0 = *$ and the map

$$X_n \to (X_1)^n$$

is a weak homotopy equivalence of spaces.

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Theorem 5. *(B) Segal monoids are equivalent to topological monoids.*

How can we get inverses?

Where $\mathbf{\Delta}^{op}$ had objects

$$[n] = \{0 \to 1 \to \cdots \to n\},$$

we define the category $\mathbf{I\Delta}^{op}$ to have objects

$$I[n] = \{0 \leftrightarrows 1 \leftrightarrows \cdots \leftrightarrows n\},$$

with appropriate morphisms.

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Definition 6. A *Segal group* is a functor $X : \mathbf{I\Delta}^{op} \to \mathcal{S}paces$ such that $X_0 = *$ and the map

$$X_n \to (X_1)^n$$

is a weak homotopy equivalence of spaces.

Theorem 7. *(B) Segal groups are equivalent to homotopy $\mathcal{T}_G$-algebras. In other words, the structure of $\mathbf{I\Delta}^{op}$ is sufficient to encode a group structure.*

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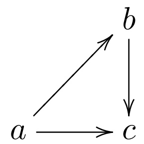
A result of Bousfield shows that we can get a group structure just using $\mathbf{\Delta}^{op}$.

These inverses can be recovered if we change our projection maps.

To get an idea of how this works, let's look at a 2-simplex.



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We use these new projections to define, for a functor $X : \mathbf{\Delta}^{op} \to \mathcal{S}paces$, a map

$$X_n \to (X_1)^n.$$

If these maps are weak homotopy equivalences, we get a group structure. We will call such functors *Bousfield groups.*

Unlike the Segal groups, the Bousfield groups can be modified nicely to obtain models for topological abelian groups.

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Thirteen ways of looking at a topological group

1. Topological groups
2. Loop spaces
3. Group-like A_∞-spaces
4. Simplicial groups
5. Strict algebras $\mathcal{T}_G \to \mathcal{S}paces$
6. Homotopy algebras $\mathcal{T}_G \to \mathcal{S}paces$

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7. Strict algebras $\mathcal{T}_G \to \mathcal{SS}ets$
8. Homotopy algebras $\mathcal{T}_G \to \mathcal{SS}ets$
9. Segal groups $\mathbf{I\Delta}^{op} \to \mathcal{S}paces$
10. Segal groups $\mathbf{I\Delta}^{op} \to \mathcal{SS}ets$
11. Bousfield groups $\mathbf{\Delta}^{op} \to \mathcal{S}paces$
12. Bousfield groups $\mathbf{\Delta}^{op} \to \mathcal{SS}ets$
13. An alternative algebraic theory approach (Badzioch-Chung-Voronov)

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