

GROUPOID CARDINALITY AND EGYPTIAN FRACTIONS

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ABSTRACT. In this paper we show that two very new questions about the cardinality of groupoids reduce to very old questions concerning the methods of ancient Egyptians for writing fractions. First, the question of whether any positive real number occurs as the groupoid cardinality of some groupoid reduces to the question of whether any positive rational number has an Egyptian fraction decomposition. Second, the question of the number of non-equivalent groupoids with a given cardinality can be answered via the number of unique Egyptian fraction decompositions.

1. GROUPS AND GROUPOIDS

The goal of this paper is to connect the very old method of Egyptian fractions with the modern idea of classifying algebraic objects. We describe an algebraic object called a groupoid, which has a well-defined cardinality given by positive rational number. Groupoids have been around for some time, but only recently has their cardinality been defined and investigated [1]. The cardinality of a groupoid has been used to define a new categorification process for describing linear algebraic objects such as Hall algebras and Hecke algebras [2]. What we are interested in here are implications for groupoid cardinality which follow from old results about Egyptian fractions decompositions.

We will work up to the definition of a groupoid by first giving a visual interpretation of a group. First, recall the definition of a group.

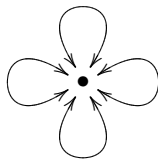
Definition 1.1. A *group* is a set equipped with an operation which is associative, has an identity element, and has inverses.

Example 1.2. The integers \mathbb{Z} form a group under addition.

Example 1.3. The group \mathbb{Z}/m of integers modulo m , also forms a group under addition.

Example 1.4. The symmetric group on n letters, S_n , has elements given by the number of ways of permuting the elements of the set $\{1, 2, \dots, n\}$. The group operation is composition of permutations.

We will think of a group using the following kind of picture.

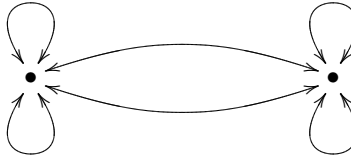


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There is one dot in the middle, and the elements of the group are drawn as the arrows. The arrows are drawn to go in both directions to indicate that each arrow has an inverse arrow that cancels it out. We can think of these arrows as “functions” from the center dot to itself. All these “functions” start and end at the same place, so we can compose them in any order.

Using this function analogy, one might ask why they all have to start and end in the same place. Instead, we can draw more general pictures, where we have more than one dot. For example one can use a diagram like the following



to visualize a groupoid.

We can think of the set of arrows as having an operation defined whenever the arrows match up (the “range dot” of one arrow is the “domain dot” of the other), and we require this operation to be associative. Each dot has an identity arrow, and all arrows have inverses with respect to these identities.

Definition 1.5. A set with a partially defined operation satisfying the above conditions is called a *groupoid*.

A groupoid may have many different *components*, or collections of dots which are not connected to one another by any arrows.

Example 1.6. An equivalence relation on a set defines a groupoid. The elements of the set form the “dots”, and there is a unique arrow from a dot to any other dot which is equivalent to it. The components of this groupoid are exactly given by the equivalence classes.

Given any dot \bullet in a groupoid, its *automorphisms* (arrows starting and ending at that dot) form a group $\text{Aut}(\bullet)$. A basic fact about groupoids is that any two dots in the same component have the same automorphism group. Therefore, we get a group associated to each component of a groupoid. Two groupoids are *equivalent* if they have the same number of components and if the automorphism group of each component of one groupoid agrees with the automorphism group of the respective component of the other.

Remark 1.7. We would be remiss if we didn’t make some reference to categories here. In the definition of a group, if we were to drop the requirement that each element have an inverse, we would have the definition of a *monoid*. Similarly, in our definition of groupoid, we could drop the requirement that each arrow have an inverse; such an object is known as a *category*. We will not need more general categories in this paper, but many of the concepts we present here can be formalized in the context of category theory.

2. GROUPOID CARDINALITY

In this section, we show how to define the cardinality of a groupoid, in a similar spirit to defining the order of a group.

Definition 2.1. The *order* of a finite group G is the number of elements of G as a set, denoted $\#G$.

Example 2.2. The group \mathbb{Z}/m has m elements, so $\#\mathbb{Z}/m = m$.

Example 2.3. The symmetric group S_n has $n!$ elements.

Example 2.4. The group of integers \mathbb{Z} is not a finite group, so we say it has infinite order.

We could define the order of a groupoid similarly, just counting the number of arrows, but we would like a more refined version for two reasons. First, we would like a definition which can be used for at least some infinite groupoids, not just finite ones. Second, we would like a definition which gives us some information about the different components of the groupoid. The following definition was developed by Baez and Dolan [1].

Definition 2.5. The *groupoid cardinality* of a groupoid G is

$$|G| = \sum_{\bullet \in G} \frac{1}{\#\text{Aut}(\bullet)},$$

when the sum is defined.

Of course, there are many groupoids for which this definition does not apply, for example if at least one of the automorphism groups is infinite. If a groupoid has infinitely many components and all the automorphism groups are finite, the cardinality is still defined if the resulting infinite series converges.

Example 2.6. If G is a group, then

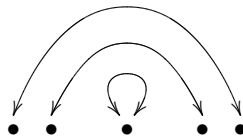
$$|G| = \frac{1}{\#G}.$$

Remark 2.7. With regard to groups, the terms *order* and *cardinality* are often used interchangeably, and our notation for groupoid cardinality is commonly used for the order of a group. Here we distinguish the two since, as the previous example shows, this definition of groupoid cardinality does not restrict to the definition of order in the special case of a group. However, it should be noticed that, for a group, the groupoid cardinality retains the same basic information, as the order has just been inverted.

Example 2.8. If G and H are groups, then their disjoint union $G \amalg H$ is a groupoid. In other words, we think of taking the groups G and H , setting them side by side, and regarding the two of them together as a two-component groupoid. The cardinality of $G \amalg H$ is

$$|G \amalg H| = \frac{1}{\#G} + \frac{1}{\#H}.$$

Example 2.9. Consider the groupoid G given by the following picture:



Recall that, in such a picture, we only draw non-trivial automorphisms at each dot. This means the two components with two dots each have a trivial automorphism group (only 1 element), and the single dot component in the middle has an automorphism group of order 2. Therefore, the groupoid cardinality is

$$|G| = 1 + 1 + \frac{1}{2} = \frac{5}{2}.$$

Looking at the definition, taking the cardinality of a groupoid must always be a positive number, since groups have positive order. But how interesting can these numbers get?

Example 2.10. Let E be the groupoid with dots the finite sets and the arrows the isomorphisms between them. This groupoid has one component for each natural number $n > 1$. The cardinality of this groupoid is

$$\begin{aligned} |E| &= \sum_{[\bullet] \in E} \frac{1}{\#\text{Aut}(\bullet)} \\ &= \sum_{n \in \mathbb{N}} \frac{1}{\#S_n} \\ &= \sum_{n \in \mathbb{N}} \frac{1}{n!} \\ &= e. \end{aligned}$$

This last example shows that groupoid cardinalities do not have to be rational numbers! This groupoid had not only an infinite number of objects, but even an infinite number of components, yet the formula for its cardinality gave a convergent series. In particular, this example illustrates that we can get more interesting things happening with groupoid cardinality than we do with group order.

One might, however, ask the following realization question.

Question 2.11. Can we get any positive real number as the cardinality of some groupoid?

Since any real number can be written as a convergent series of rational numbers (given by its decimal expansion), it suffices to determine whether we can obtain any positive rational number as a groupoid with finitely many components. We can get whole numbers n by taking n dots with no non-identity arrows. Therefore, the interesting part of the question is whether we can find a groupoid whose cardinality is q for any rational number $0 < q < 1$.

Any rational number of the form $\frac{1}{n}$ is easy, since we can just take the groupoid cardinality of any group of order n , for example \mathbb{Z}/n . Furthermore, any rational number of the form $\frac{1}{n} + \frac{1}{m}$ is also easy, since we can use a disjoint union of groups such as $\mathbb{Z}/n \amalg \mathbb{Z}/m$. In fact, for any rational number $\frac{m}{n}$, we can take

$$|\mathbb{Z}/n \amalg \cdots \amalg \mathbb{Z}/n|,$$

where there are m copies of \mathbb{Z}/n . So the answer to our question is yes.

However, this last step is not entirely satisfying, since we have just repeated the same group over and over again. We could instead ask if there are more interesting ways to obtain such a groupoid. For example, could we instead find a groupoid

with a given cardinality, such that no two components of the groupoid have the same cardinality? In other words, can we write any rational number between 0 and 1 in the form

$$\frac{m}{n} = \sum_i \frac{1}{n_i}$$

where all the positive integers n_i are distinct? This question leads us to the ancient study of Egyptian fractions.

3. EGYPTIAN FRACTION DECOMPOSITIONS

Much of what we know about ancient Egyptian mathematics came from the discovery of the Rhind Papyrus in the 1850s [5]. Our interest here is concerned with their curious way of working with fractions.

The ancient Egyptians only had a notation for fractions of the form $\frac{1}{n}$, not for more general ones such as $\frac{7}{12}$. Furthermore, when they needed to write more complicated rational numbers, they did so by taking sums of these unit fractions where no summands were repeated. From our perspective, the question arises whether the Egyptians were limited by this method. Given any rational number q with $0 < q < 1$, can it be written as the sum of finitely many distinct unit fractions? We shall refer to a such a sum as an *Egyptian fraction decomposition*.

Example 3.1. This idea might sound cumbersome, but it can actually be useful in practice. As an elementary example, suppose you have 5 muffins that you want to divide among 8 people. You could measure them all carefully and give each person $\frac{5}{8}$ of a muffin, but it would be far easier to give each person $\frac{1}{2}$ of a muffin and then divide the remaining muffin into 8 pieces, so each person gets $\frac{5}{8} = \frac{1}{2} + \frac{1}{8}$ of a muffin.

Notice that Egyptian fraction decompositions are not unique. For example

$$\frac{3}{4} = \frac{1}{2} + \frac{1}{4}$$

but also

$$\frac{3}{4} = \frac{1}{2} + \frac{1}{5} + \frac{1}{20}.$$

We get the second equation by applying the “splitting algorithm” which says

$$\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)}.$$

The following theorem answers our question.

Theorem 3.2. *Every rational number has an Egyptian fraction decomposition.*

In fact, we have an even stronger result.

Theorem 3.3. *Every rational number has infinitely many distinct Egyptian fraction decompositions.*

Let us first look at the main idea behind the proof of Theorem 3.3. The important fact that we are going to use is the Egyptian fraction decomposition for 1 given by

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}.$$

This sum is given by applying the splitting algorithm twice to the number 1. To see how this equation can be applied, we return to our example of

$$\frac{3}{4} = \frac{1}{2} + \frac{1}{4}.$$

We can divide the first equation by 4 to get

$$\frac{1}{4} = \frac{1}{4} \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6} \right) = \frac{1}{8} + \frac{1}{12} + \frac{1}{24}.$$

Hence, we can replace $\frac{1}{4}$ in our original decomposition with its expansion given here to obtain

$$\frac{3}{4} = \frac{1}{2} + \frac{1}{8} + \frac{1}{12} + \frac{1}{24}.$$

But then we can go back to our expression for 1 and divide it by 24, obtaining

$$\frac{1}{24} = \frac{1}{48} + \frac{1}{72} + \frac{1}{144}$$

leading to a still longer decomposition

$$\frac{3}{4} = \frac{1}{2} + \frac{1}{8} + \frac{1}{12} + \frac{1}{48} + \frac{1}{72} + \frac{1}{144}.$$

Further decomposing each final term leads to an infinite number of Egyptian fraction decompositions for a given positive rational number, once we establish via Theorem 3.2 that at least one exists. We now formalize the argument from this example into a proof.

Proof of Theorem 3.3. Let q be a rational number and let

$$q = \frac{1}{d_1} + \cdots + \frac{1}{d_n}$$

be an Egyptian fraction decomposition, so that each d_i is distinct. Further assume that d_1, \dots, d_n is an increasing sequence of integers. Then use the decomposition for 1 used above to observe that

$$\frac{1}{d_n} = \frac{1}{2d_n} + \frac{1}{3d_n} + \frac{1}{6d_n}.$$

Then

$$q = \frac{1}{d_1} + \cdots + \frac{1}{d_{n-1}} + \frac{1}{2d_n} + \frac{1}{3d_n} + \frac{1}{6d_n}$$

is another decomposition. Because $d_n < 2d_n < 3d_n < 6d_n$, the sequence

$$d_1, \dots, d_{n-1}, 2d_n, 3d_n, 6d_n$$

is still an increasing sequence; in particular, we have guaranteed that we have not repeated any denominators. Therefore, given any Egyptian fraction decomposition, we can find another distinct one. \square

We now look at an example to illustrate one approach to proving Theorem 3.2, as given in [3]. Suppose that $\frac{m}{n}$ is a fraction with $1 < m < n$. The strategy, which goes back to Fibonacci [4], is to find the largest possible unit fraction smaller than $\frac{m}{n}$ and show that the process of doing so eventually terminates. Since we are always taking the largest possible Egyptian fraction less than our given rational number, this procedure is called the *greedy algorithm*.

For example, consider $\frac{4}{13}$. We first notice that

$$\frac{1}{4} = \frac{4}{16} < \frac{4}{13} < \frac{4}{12} = \frac{1}{3}$$

so $\frac{1}{4}$ is the smallest unit fraction less than $\frac{4}{13}$. We then take the difference and obtain that

$$\frac{4}{13} = \frac{1}{4} + \frac{3}{52}.$$

Since the second fraction in this sum is not of Egyptian type, we repeat the procedure and see that

$$\frac{1}{18} = \frac{3}{54} < \frac{3}{52} < \frac{3}{51} = \frac{1}{17}$$

and subtract to see that

$$\frac{4}{13} = \frac{1}{4} + \frac{1}{18} + \frac{1}{468}.$$

To show that such a process always works, we need to prove that it must terminate, giving a finite number of Egyptian fractions. Again, our argument of proof is taken from [3].

Proof of Theorem 3.2. For any $\frac{m}{n}$, write

$$\frac{1}{d_1} < \frac{m}{n} < \frac{1}{d_1 - 1}.$$

In other words, we know that

$$\frac{m}{n} = \frac{1}{d_1} + \left(\frac{m}{n} - \frac{1}{d_1} \right) = \frac{1}{d_1} + \frac{md_1 - n}{nd_1}.$$

The inequality

$$\frac{m}{n} < \frac{1}{d_1 - 1}$$

implies that

$$m < \frac{n}{d_1 - 1},$$

or

$$m(d_1 - 1) < n.$$

We can multiply to get

$$md_1 - m < n$$

and rearrange to see that

$$md_1 - n < m.$$

In other words, the numerator $md_1 - n$ is smaller than our original numerator m . We can then find a positive integer d_2 such that

$$\frac{1}{d_2} < \frac{md_1 - n}{nd_1} < \frac{1}{d_2 - 1}$$

which gives a remainder whose numerator is strictly smaller than $md_1 - n$. Since each numerator is a positive integer strictly smaller than the last, the process must eventually give a numerator of 1, terminating the algorithm. \square

Notice, however, that the greedy decomposition may not necessarily be the shortest Egyptian fraction decomposition. For example, applying the greedy algorithm to $\frac{83}{140}$ yields

$$\frac{83}{140} = \frac{1}{2} + \frac{1}{11} + \frac{1}{514} + \frac{1}{395780}.$$

However, this fraction can also be decomposed as

$$\frac{83}{140} = \frac{1}{4} + \frac{1}{5} + \frac{1}{7}.$$

A calculator for computing *shortest* Egyptian fraction decompositions can be found at [3].

4. IMPLICATIONS FOR GROUPOID CARDINALITY

The results of the previous section immediately imply two interesting facts about groupoid cardinality. First, the existence of Egyptian fraction decompositions as given by Theorem 3.2 gives us an interesting groupoid with this number as the cardinality.

Theorem 4.1. *Any positive rational number is the cardinality of a groupoid with no two components having the same cardinality.*

The second result follows from Theorem 3.3.

Theorem 4.2. *Any positive rational number is the cardinality of infinitely many distinct such groupoids.*

Not only can we find infinitely many groupoids with the same cardinality, but we can choose them so that no two are equivalent to one another.

Both of these results have interesting implications to research. For instance, when working with the concept of groupoid cardinality, it is typical to have a desired cardinality in mind, then work backward to find a groupoid which has this value as its cardinality. As an example, we already saw that the groupoid E of finite sets and bijections has a groupoid cardinality of e . We got this cardinality by considering the power series expansion for e^x :

$$e^x = \sum_n \frac{x^n}{n!}$$

with $x = 1$. What if, instead, we took other values for x ? For example, we might want a groupoid with cardinality e^2 , and so we could start with the power series expansion:

$$e^2 = \sum_n \frac{2^n}{n!}.$$

The first thing to note is that this sum no longer uses just unit fractions. We could then try to write different decompositions for each fraction $\frac{2^n}{n!}$ until we find something that looks familiar. The eventual goal is to then find a groupoid that is more interesting than simply disjoint unions of groups.

Also, the second result is in stark contrast to the theory of finite groups, where one of the great recent achievements was the classification of finite simple groups. Thanks to Theorem 4.2, it would be expected that any kind of classification for finite groupoids would be still more intricate.

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