

HOMOTOPY FIBER PRODUCTS OF HOMOTOPY THEORIES

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ABSTRACT. Given an appropriate diagram of left Quillen functors between model categories, one can define a notion of homotopy fiber product, but one might ask if it is really the correct one. Here, we show that this homotopy pullback is well-behaved with respect to translating it into the setting of more general homotopy theories, given by complete Segal spaces, where we have well-defined homotopy pullbacks.

1. INTRODUCTION

Homotopy theory originates with the study of topological spaces up to weak homotopy equivalence, where two spaces are weakly homotopy equivalent if there is a map between them inducing isomorphisms on all the homotopy groups. The question of whether the nice properties of topological spaces held in other settings led to the development of the notion of a model category by Quillen [22]. The study of model categories is then a more abstract form of homotopy theory, one which has been investigated extensively.

One could then consider model categories themselves as objects of a category and consider Quillen equivalences as weak equivalences between them. In this framework, one could ask questions about relationships between model categories; for example, what would a homotopy limit or homotopy colimit of a diagram of model categories be? Unfortunately, there are no immediate answers to these questions because at present there is no known model structure on the category of model categories. In one particular case of a homotopy limit, we have a plausible construction, that of the homotopy fiber product. This construction was first explained to the author by Smith; it appears in the literature in a paper of Toën [27], where it is useful in his development of derived Hall algebras. One might ask whether, without a model category of model categories, we can really accept this definition as a genuine homotopy pullback construction.

However, one can also consider a homotopy theory to be something more general than a model category, such as a category with a specified class of weak equivalences, or maps one would like to consider as equivalences but which are not necessarily isomorphisms. While there may or may not be a model structure on such a category, one can heuristically think of formally inverting the weak equivalences, set-theoretic problems notwithstanding. Furthermore, such “homotopy theories” can be regarded as the objects of a model category.

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In a series of papers [7], [9], [10], Dwyer and Kan develop the theory of simplicial localizations, which are simplicial categories corresponding to model categories, or, more generally, categories with specified weak equivalences. Since every such category has a simplicial localization, and since, up to a natural notion of equivalence of simplicial categories, every simplicial category arises as the simplicial localization of some category with weak equivalences [8, 2.5], simplicial categories can be considered to be models for homotopy theories. With these weak equivalences, often called Dwyer-Kan equivalences, the category of (small) simplicial categories can itself be considered as a homotopy theory, and in fact it has a model structure [3]. Thus, within this framework one can address questions about homotopy theories that are natural to ask in a given model category, such as characterizations of homotopy limits and colimits.

Unfortunately, the model structure on the category of simplicial categories is technically difficult to work with. The weak equivalences are particularly challenging to identify, for example. Fortunately, we have a choice of several other equivalent model categories in which to address these questions. Work of the author and of Joyal and Tierney has proved that the complete Segal space model structure on the category of simplicial spaces, two different Segal category model structures on the category of Segal precategories, and the quasi-category model structure on the category of simplicial sets are all equivalent as model categories, and thus each is a model for the homotopy theory of homotopy theories [4], [17], [19].

In this paper, we address the construction of the homotopy fiber product of model categories and its analogue within the complete Segal space model structure. Of the various models mentioned above, the complete Segal space model structure is the best setting in which to answer this question due to the particularly nice description of the relevant weak equivalences.

Some clarification of terminology is needed here, as there are actually two notions of what is meant by a homotopy fiber product of model categories. One is more restrictive than the other; the stronger is the one we would expect to correspond to a homotopy pullback, but the weaker is the one which can be given the structure of a model category. We show that at least in some cases this model structure can be localized so that the fibrant-cofibrant objects satisfy the more restrictive condition.

The main result of this paper is that the stricter definition does in fact correspond to a homotopy pullback when we work in the complete Segal space setting. We also give a characterization of the complete Segal spaces arising from the less restrictive description.

We should point out that this construction has been used in special cases, for example by Hüttemann, Klein, Vogell, Waldhausen, and Williams in [15], and as an example of more general constructions, for example the twisted diagrams of Hüttemann and Röndigs [16], and the model categories of comma categories as given by Stanculescu [26].

It is also important to note that translating this question into the setting of more general homotopy theories is not merely a temporary solution until one finds a model category of model categories. In practice, model structures are often hard to establish, and furthermore, the condition of having a Quillen pair between two such model structures is a rigid one. Being able to consider homotopy theories as more flexible kinds of objects, and having morphisms between them less structured, makes it more likely that we can actually implement such a construction. We

consider such a case in Example 3.3. Yet, with the relationship between the two settings established, we can use the additional structure when we do indeed have it.

In fact, part of our motivation for making the comparison in this paper is to generalize Toën’s development of derived Hall algebras. Where he defines an associative algebra corresponding to stable model categories given by modules over a dg category, we would like to define such an algebra using a more general stable homotopy theory, namely, one given by a stable complete Segal space. The main result of this paper allows us to use a homotopy pullback of complete Segal spaces in the setting where Toën uses a homotopy fiber product of model categories.

The dual notion of homotopy pushouts of model categories, as well as more general homotopy limits and homotopy colimits, will be considered in later work.

2. MODEL CATEGORIES AND MORE GENERAL HOMOTOPY THEORIES

In this section we give a brief review of model categories and their relationship with the complete Segal space model for more general homotopy theories.

Recall that a *model category* \mathcal{M} is a category with three distinguished classes of morphisms: weak equivalences, fibrations, and cofibrations, satisfying five axioms [11, 3.3]. Given a model category structure, one can pass to the homotopy category $\mathrm{Ho}(\mathcal{M})$, in which the weak equivalences from \mathcal{M} become isomorphisms. In particular, the weak equivalences, as the morphisms that we wish to invert, make up the most important part of a model category. An object x in \mathcal{M} is *fibrant* if the unique map $x \rightarrow *$ to the terminal object is a fibration. Dually, an object x in \mathcal{M} is *cofibrant* if the unique map $\phi \rightarrow x$ from the initial object is a cofibration.

Given a model category \mathcal{M} , there is also a model structure on the category $\mathcal{M}^{[1]}$, often called the morphism category of \mathcal{M} . The objects of $\mathcal{M}^{[1]}$ are morphisms of \mathcal{M} , and the morphisms of $\mathcal{M}^{[1]}$ are given by pairs of morphisms making the appropriate square diagram commute. A morphism in $\mathcal{M}^{[1]}$ is a weak equivalence (or cofibration) if its component maps are weak equivalences (or cofibrations) in \mathcal{M} . More generally, $\mathcal{M}^{[n]}$ is the category with objects strings of n composable morphisms in \mathcal{M} ; the model structure can be defined analogously.

One could, more generally, consider categories with weak equivalences and no additional structure, and then formally invert the weak equivalences. This process does give a homotopy category, but it frequently has the disadvantage of having a proper class of morphisms between any two given objects. If we are willing to accept such set-theoretic problems, then we can work in this situation; the advantage of a model structure is that it provides enough additional structure so that we can take homotopy classes of maps and hence avoid these difficulties. In this paper, we will use both notions of a “homotopy theory,” depending on the circumstances.

We would, however, like to work with nice objects modeling categories with weak equivalences. While there are several options, the model that we use in this paper is that of complete Segal spaces.

Recall that the simplicial category Δ^{op} is defined to be the category with objects finite ordered sets $[n] = \{0 \rightarrow 1 \rightarrow \cdots \rightarrow n\}$ and morphisms the opposites of the order-preserving maps between them. A *simplicial set* is then a functor

$$K: \Delta^{op} \rightarrow \mathit{Sets}.$$

We denote by $\mathcal{S}Sets$ the category of simplicial sets, and this category has a natural model category structure equivalent to the standard model structure on topological spaces [12, I.10].

One can just as easily consider more general simplicial objects; in this paper we consider *simplicial spaces* (also called bisimplicial sets), or functors

$$X: \Delta^{op} \rightarrow \mathcal{S}Sets.$$

There are several model category structures on the category of bisimplicial sets. An important one is the Reedy model structure [23], which is equivalent to the injective model structure, where the weak equivalences are given by levelwise weak equivalences of simplicial sets, and the cofibrations are given likewise [13, 15.8.7]. Given a simplicial set K , we also denote by K the simplicial space which has the simplicial set K at every level. We denote by K^t , or “ K -transposed”, the constant simplicial space in the other direction, where $(K^t)_n = K_n$, where on the right-hand side K_n is regarded as a discrete simplicial set.

We use the idea, originating with Dwyer and Kan, that a simplicial category, or category enriched over simplicial sets, models a homotopy theory, in the following way. Using either of their two notions of simplicial localization, one can obtain from a category with weak equivalences a simplicial category [9], [10], and there is a model structure $\mathcal{S}C$ on the category of all small simplicial categories [3]; thus, we have a so-called homotopy theory of homotopy theories. One useful consequence of taking the simplicial category corresponding to a model category is that we can use it to describe *homotopy function complexes*, or homotopy-invariant mapping spaces $\text{Map}^h(x, y)$ between objects of a model category which is not necessarily equipped with the additional structure of a simplicial model category. We use, in particular, the simplicial set $\text{Aut}^h(x)$ of homotopy invertible self-maps of an object x .

Here we also consider simplicial spaces satisfying conditions imposing a notion of composition up to homotopy. These Segal spaces and complete Segal spaces were first introduced by Rezk [24], and the name is meant to be suggestive of similar ideas first presented by Segal [25].

Definition 2.1. [24, 4.1] A *Segal space* is a Reedy fibrant simplicial space W such that the Segal maps

$$\varphi_n: W_n \rightarrow \underbrace{W_1 \times_{W_0} \cdots \times_{W_0} W_1}_n$$

are weak equivalences of simplicial sets for all $n \geq 2$.

Given a Segal space W , we can consider its “objects” $\text{ob}(W) = W_{0,0}$, and, between any two objects x and y , the “mapping space” $\text{map}_W(x, y)$, given by the homotopy fiber of the map $W_1 \rightarrow W_0 \times W_0$ given by the two face maps $W_1 \rightarrow W_0$. The Segal condition given here tells us that a Segal space has a notion of n -fold composition of mapping spaces, at least up to homotopy. More precise constructions are given by Rezk [24, §5]. Using this composition, we can define “homotopy equivalences” in a natural way, and then speak of the subspace of W_1 whose components contain homotopy equivalences, denoted W_{hoequiv} . Notice that the degeneracy map $s_0: W_0 \rightarrow W_1$ factors through W_{hoequiv} ; hence we may make the following definition.

Definition 2.2. [24, §6] A *complete Segal space* is a Segal space W such that the map $W_0 \rightarrow W_{\text{hoequiv}}$ is a weak equivalence of simplicial sets.

Given this definition, we can describe the model structure on the category of simplicial spaces that is used throughout this paper.

Theorem 2.3. [24, §7] *There is a model category structure \mathcal{CSS} on the category of simplicial spaces, obtained as a localization of the Reedy model structure such that:*

- (1) *the fibrant objects are the complete Segal spaces,*
- (2) *all objects are cofibrant, and*
- (3) *the weak equivalences between complete Segal spaces are levelwise weak equivalences of simplicial sets.*

Now we return to the idea that a complete Segal space models a homotopy theory.

Theorem 2.4. [4] *The model categories \mathcal{SC} and \mathcal{CSS} are Quillen equivalent.*

Furthermore, due to work of Rezk [24] which was continued by the author [2], we can actually characterize, up to weak equivalence, the complete Segal space arising from a simplicial category, or more specifically, from a model category. Rezk defines a functor which we denote L_C from the category of model categories to the category of simplicial spaces; given a model category \mathcal{M} , we have that

$$L_C(\mathcal{M})_n = \text{nerve}(\text{we}(\mathcal{M}^{[n]})).$$

Here, $\mathcal{M}^{[n]}$ is defined as above, and $\text{we}(\mathcal{M}^{[n]})$ denotes the subcategory of $\mathcal{M}^{[n]}$ whose morphisms are the weak equivalences. While the resulting simplicial space is not in general Reedy fibrant, and hence not a complete Segal space, Rezk proves that taking a Reedy fibrant replacement is sufficient to obtain a complete Segal space [24, 8.3]. Hence, for the rest of this paper we assume that the functor L_C includes composition with this Reedy fibrant replacement and therefore gives a complete Segal space.

One other difficulty that arises in this definition is the fact that it is only a well-defined functor on the category whose objects are model categories and whose morphisms preserve weak equivalences. This restriction on the morphisms is stronger than we would like; it would be preferable to have such a functor defined on the category of model categories with morphisms Quillen functors, where weak equivalences are only preserved between either fibrant or cofibrant objects. To obtain this more suitable functor, we consider \mathcal{M}^c , the full subcategory of \mathcal{M} whose objects are cofibrant. While \mathcal{M}^c may no longer have the structure of a model category, it is still a category with weak equivalences, and therefore Rezk's definition can still be applied to it, so our new definition is

$$L_C(\mathcal{M})_n = \text{nerve}(\text{we}((\mathcal{M}^c)^{[n]})).$$

Each space in this new diagram is weakly equivalent to the one given by the previous definition, and now the construction is functorial on the category of model categories with morphisms the left Quillen functors.

If one wanted to consider right Quillen functors instead, we could take the full subcategory of fibrant objects, \mathcal{M}^f , rather than \mathcal{M}^c .

Before stating the theorem giving the characterization, we give some facts about simplicial monoids, or functors from $\mathbf{\Delta}^{op}$ to the category of monoids. Given a simplicial monoid M (or, more commonly, a simplicial group), we can find a classifying complex of M , a simplicial set whose geometric realization is the classifying space BM . A precise construction can be made for this classifying space by the \overline{WM}

construction [12, V.4.4], [21]. As we are not so concerned here with the precise construction as with the fact that such a classifying space exists, we will simply write BM for the classifying complex of M .

Theorem 2.5. [2, 7.3] *Let \mathcal{M} be a model category. For x an object of \mathcal{M} denote by $\langle x \rangle$ the weak equivalence class of x in \mathcal{M} , and denote by $\text{Aut}^h(x)$ the simplicial monoid of self weak equivalences of x . Up to weak equivalence in the model category \mathcal{CSS} , the complete Segal space $L_C(\mathcal{M})$ looks like*

$$\coprod_{\langle x \rangle} B\text{Aut}^h(x) \Leftarrow \coprod_{\langle \alpha: x \rightarrow y \rangle} B\text{Aut}^h(\alpha) \Leftarrow \cdots .$$

(We should point out that the reference (Theorem 7.3 of [2]) gives a characterization of the complete Segal space arising from a simplicial category, not from a model category. However, the results of §6 of that same paper allow one to translate it to the theorem as stated here.)

This characterization, together with the fact that weak equivalences between complete Segal spaces are levelwise weak equivalences of simplicial sets, enables us to compare complete Segal spaces arising from different model categories.

3. HOMOTOPY FIBER PRODUCTS OF MODEL CATEGORIES

We begin with the definition of homotopy fiber product as given by Toën in [27]. First, suppose that

$$\mathcal{M}_1 \xrightarrow{F_1} \mathcal{M}_3 \xleftarrow{F_2} \mathcal{M}_2$$

is a diagram of left Quillen functors of model categories. Define their *homotopy fiber product* to be the model category $\mathcal{M} = \mathcal{M}_1 \times_{\mathcal{M}_3}^h \mathcal{M}_2$ whose objects are given by 5-tuples $(x_1, x_2, x_3; u, v)$ such that each x_i is an object of \mathcal{M}_i fitting into a diagram

$$F_1(x_1) \xrightarrow{u} x_3 \xleftarrow{v} F_2(x_2).$$

A morphism of \mathcal{M} , say $f: (x_1, x_2, x_3; u, v) \rightarrow (y_1, y_2, y_3; z, w)$, is given by a triple of maps $f_i: x_i \rightarrow y_i$ for $i = 1, 2, 3$, such that the following diagram commutes:

$$\begin{array}{ccccc} F_1(x_1) & \xrightarrow{u} & x_3 & \xleftarrow{v} & F_2(x_2) \\ \downarrow F_1(f_1) & & \downarrow f_3 & & \downarrow F_2(f_2) \\ F_1(y_1) & \xrightarrow{z} & y_3 & \xleftarrow{w} & F_2(y_2). \end{array}$$

This category \mathcal{M} can be given the structure of a model category, where the weak equivalences and cofibrations are given levelwise. In other words, f is a weak equivalence (or cofibration) if each map f_i is a weak equivalence (or cofibration) in \mathcal{M}_i .

A more restricted definition of this construction requires that the maps u and v be weak equivalences in \mathcal{M}_3 . Unfortunately, if we impose this additional condition, the resulting category cannot be given the structure of a model category because it is not closed under limits and colimits. However, intuition suggests that we really want to require u and v to be weak equivalences in order to get an appropriate homotopy pullback. We would like to have a localization of the model structure on \mathcal{M} described above such that the fibrant-cofibrant objects have the maps u and v weak equivalences. In at least some situations, we can find such a localization.

Recall that a model category is *combinatorial* if it is cofibrantly generated and locally presentable as a category [5, 2.1].

Theorem 3.1. *Let \mathcal{M} be the homotopy fiber product of a diagram of left Quillen functors*

$$\mathcal{M}_1 \xrightarrow{F_1} \mathcal{M}_3 \xleftarrow{F_2} \mathcal{M}_2$$

where each of the categories \mathcal{M}_i is combinatorial. Further assume that \mathcal{M} is right proper. Then there exists a right Bousfield localization of \mathcal{M} whose fibrant and cofibrant objects $(x_1, x_2, y_2; u, v)$ have both u and v weak equivalences in \mathcal{M}_3 .

Proof. Since the categories \mathcal{M}_1 , \mathcal{M}_2 , and \mathcal{M}_3 are combinatorial, and hence locally presentable, we can find, for each $i = 1, 2, 3$, a set \mathcal{A}_i of objects of \mathcal{M}_i which generates all of \mathcal{M}_i by filtered colimits [5, 2.2]. Furthermore, we can assume that the objects of \mathcal{A}_i are all cofibrant. (An explicit such set can be found, for example, using Dugger's notion of a presentation of a combinatorial model category [6].) Given $a_1 \in \mathcal{A}_1$ and $a_2 \in \mathcal{A}_2$, consider the class of all objects x_3 such that there are pairs of weak equivalences

$$F_1(a_1) \xrightarrow{\simeq} x_3 \xleftarrow{\simeq} F_2(a_2).$$

Since \mathcal{A}_1 and \mathcal{A}_2 are sets, we can choose one representative of x_3 for each pair a_1 and a_2 with $F_1(a_1)$ weakly equivalent to $F_2(a_2)$. Taking the union of this set together with the generating set \mathcal{A}_3 for \mathcal{M}_3 , we obtain a set which we denote \mathcal{B}_3 . For $i = 1, 2$, let $\mathcal{B}_i = \mathcal{A}_i$.

In \mathcal{M} , consider the following set of objects:

$$\{(x_1, x_2, x_3; u, v) \mid x_i \in \mathcal{B}_i, u, v \text{ weak equivalences in } \mathcal{M}_3\}.$$

By taking filtered colimits, we can obtain from this set all objects $(x_1, x_2, x_3; u, v)$ of \mathcal{M} for which the maps u and v are weak equivalences; while arbitrary colimits do not necessarily preserve these weak equivalences, filtered colimits do [5, 7.3]. Thus, we can take a right Bousfield localization of \mathcal{M} with respect to this set of objects; if \mathcal{M} is right proper, then this localization has a model structure [13, 5.1.1], [1]. \square

Unfortunately, it seems to be difficult to describe conditions on the model categories \mathcal{M}_1 , \mathcal{M}_2 , and \mathcal{M}_3 guaranteeing that \mathcal{M} is right proper. We can weaken this condition somewhat, using a remark of Hirschhorn [13, 5.1.2]. Alternatively, Barwick discusses the structure which is retained after taking a right Bousfield localization of a model category which is not necessarily right proper [1]. Nonetheless, when the conditions of this theorem are not satisfied, we can still use the original levelwise model structure on \mathcal{M} and simply restrict to the appropriate subcategory when we want to require u and v to be weak equivalences.

In order to determine whether this construction really gives a homotopy fiber product of homotopy theories, we need to translate it into the complete Segal space model structure via the functor L_C . When we require the maps u and v to be weak equivalences, we can still take the associated complete Segal space even without a model structure, and we do get a homotopy pullback in the model category \mathcal{CSS} . The proof of this statement is given in the next section. However, the more general construction also has a precise description as well, which we give in the following section.

We conclude this section with a few examples.

Example 3.2. We begin with some comments on the use of homotopy fiber products of model categories as used by Toën to prove associativity of his derived Hall algebras [27]. In this situation, we have a stable model category; this extra assumption that the homotopy category is triangulated implies that our model category has a “zero object” so that the initial and terminal objects coincide. We denote this object 0.

Let T be a dg category, or category enriched over chain complexes over a finite field k . Then a dg module over T is a dg functor $T \rightarrow C(k)$, where $C(k)$ denotes the category of chain complexes of modules over k . There is a model structure $\mathcal{M}(T)$ on the category of such modules over a fixed T , where the weak equivalences and fibrations are given levelwise [28, §3].

Given an object of $\mathcal{M}(T)^{[1]}$, namely a map $f: x \rightarrow y$, let $F: \mathcal{M}(T)^{[1]} \rightarrow \mathcal{M}(T)$ be the target map, so that $F(f: x \rightarrow y) = y$. Let $C: \mathcal{M}(T)^{[1]} \rightarrow \mathcal{M}(T)$ be the cone map, so that $C(f: x \rightarrow y) = y \amalg_x 0$. Using these functors, we get a diagram

$$\begin{array}{ccc} & \mathcal{M}(T)^{[1]} & \\ & \downarrow C & \\ \mathcal{M}(T)^{[1]} & \xrightarrow{F} & \mathcal{M}(T). \end{array}$$

To understand the homotopy fiber product \mathcal{M} of this diagram, Toën uses the model structure on the homotopy fiber product given by levelwise maps; eventually in the proof he adds the additional assumption that the maps u and v in the definition be weak equivalences [27, §4]. The homotopy fiber product \mathcal{M} given by this diagram is equivalent to the model category $\mathcal{M}(T)^{[2]}$ whose objects are pairs of composable morphisms in $\mathcal{M}(T)$.

Example 3.3. Here we consider the following special case of a homotopy pullback, the homotopy fiber of a map. Therefore, this definition of homotopy fiber product of model categories leads to the following definition.

Definition 3.4. Let $F: \mathcal{M} \rightarrow \mathcal{N}$ be a left Quillen functor of model categories. Then the *homotopy fiber* of F is the homotopy fiber product of the diagram

$$\begin{array}{ccc} & \mathcal{M} & \\ & \downarrow F & \\ * & \longrightarrow & \mathcal{N} \end{array}$$

where the map $* \rightarrow \mathcal{N}$ is necessarily the map from the trivial model category to the initial object ϕ of \mathcal{N} .

Using our definition, the objects of this homotopy fiber are triples $(*, m, n; u, v)$, where $*$ denotes the single object of the trivial model category $*$, m is an object of \mathcal{M} , n is an object of \mathcal{N} , $u: \phi \rightarrow n$ is the unique such map, and $v: F(m) \rightarrow n$. Imposing our condition that u and v be weak equivalences, we get that n must be weakly equivalent to the initial object of \mathcal{N} , and m is any object of \mathcal{M} whose image under F is weakly equivalent to the initial object of \mathcal{N} .

While this definition follows naturally from the usual notions, it is unsatisfactory for many purposes. The requirement that the functors in the pullback diagram be left Quillen is a very rigid one. One might perhaps prefer to look at the homotopy fiber over some other object, but here one cannot.

Example 3.5. A further specialization of this definition illustrates its particularly odd nature. If we take the analogue of a loop space and define the “loop model category” as the homotopy pullback of the diagram

$$\begin{array}{ccc} & * & \\ & \downarrow & \\ * & \longrightarrow & \mathcal{M} \end{array}$$

for any model category \mathcal{M} , we simply get the subcategory of \mathcal{M} whose objects are weakly equivalent to the initial object.

4. HOMOTOPY PULLBACKS OF COMPLETE SEGAL SPACES

Consider the functor L_C which takes a model category (or simplicial category) to a complete Segal space. Given a homotopy fiber square of model categories as defined in the previous section (namely, where we require the maps u and v to be weak equivalences), we can apply this functor to obtain a homotopy commutative square

$$\begin{array}{ccc} L_C\mathcal{M} & \longrightarrow & L_C\mathcal{M}_2 \\ \downarrow & & \downarrow \\ L_C\mathcal{M}_1 & \longrightarrow & L_C\mathcal{M}_3. \end{array}$$

Alternatively, we could apply the functor L_C only to the original diagram and take the homotopy pullback, which we denote P , and obtain the following diagram:

$$\begin{array}{ccc} P & \longrightarrow & L_C\mathcal{M}_2 \\ \downarrow & & \downarrow \\ L_C\mathcal{M}_1 & \longrightarrow & L_C\mathcal{M}_3. \end{array}$$

Since P is a homotopy pullback, there exists a natural map $L_C\mathcal{M} \rightarrow P$.

Theorem 4.1. *The map*

$$L_C\mathcal{M} \rightarrow P = L_C\mathcal{M}_1 \times_{L_C\mathcal{M}_3}^h L_C\mathcal{M}_2$$

is a weak equivalence of complete Segal spaces.

To prove this theorem, we would like to be able to use Theorem 2.5 which characterizes the complete Segal spaces that result from applying the functor L_C to a model category. However, this theorem only gives the homotopy type of each space in the simplicial diagram, not an explicit description of the precise spaces we obtain. Thus, we begin by unpacking this characterization in order to obtain actual maps between these “nice” versions of the complete Segal spaces $L_C\mathcal{M}_i$. More details can be found in [2, §7], where Theorem 2.5 is proved.

We begin with the description of the space at level zero. Given a model category \mathcal{M} , we can take its corresponding simplicial category $L\mathcal{M}$ given by Dwyer-Kan simplicial localization. Denote by \mathcal{C} the sub-simplicial category of $L\mathcal{M}$ whose morphisms are all invertible up to homotopy. Then there is a weak equivalence $F(\mathcal{C}) \rightarrow \mathcal{C}$, where $F(\mathcal{C})$ denotes the free simplicial category on \mathcal{C} [10, §2]. Then taking a groupoid completion of $F(\mathcal{C})$ gives a simplicial groupoid $F(\mathcal{C})^{-1}F(\mathcal{C})$. The characterization of the corresponding complete Segal space uses the fact that this

simplicial groupoid is equivalent to one which is the disjoint union of simplicial groups, say \mathcal{G} . There is a functor $F(\mathcal{C})^{-1}F(\mathcal{C}) \rightarrow \mathcal{G}$ collapsing each component down to one with a single object. Thus, we obtain a zig-zag of weak equivalences of simplicial categories

$$\mathcal{G} \leftarrow F(\mathcal{C})^{-1}F(\mathcal{C}) \leftarrow F(\mathcal{C}) \rightarrow \mathcal{C}.$$

Since all these constructions are functorial, by using Theorem 2.5, we are essentially passing from working with \mathcal{C} to working with \mathcal{G} .

We can apply these same constructions to the morphism category $\mathcal{M}^{[1]}$ to understand the space at level one, and, more generally, to $\mathcal{M}^{[n]}$ to obtain the description of the space at level n .

Proof of Theorem 4.1. Since all the objects in question are complete Segal spaces, i.e., local objects in the model structure $\mathcal{CS}\mathcal{S}$, it suffices to show that the map $L_C\mathcal{M} \rightarrow P$ is a levelwise weak equivalence of simplicial sets. Let us begin by comparing the space at level zero for each. The space P_0 looks like

$$(L_C\mathcal{M}_1)_0 \times_{(L_C\mathcal{M}_3)_0}^h (L_C\mathcal{M}_2)_0 = \coprod_{\langle x_1 \rangle} B\text{Aut}^h(x_1) \times_{\coprod_{\langle x_3 \rangle} B\text{Aut}^h(x_3)}^h \coprod_{\langle x_2 \rangle} B\text{Aut}^h(x_2).$$

On the other hand, $(L_C\mathcal{M})_0$ looks like

$$\coprod_{\langle (x_1, x_2, x_3; u, v) \rangle} B\text{Aut}^h((x_1, x_2, x_3; u, v)).$$

However, since the classifying space functor B commutes with taking the disjoint union, this space is equivalent to

$$B \left(\coprod_{\langle (x_1, x_2, x_3; u, v) \rangle} \text{Aut}^h((x_1, x_2, x_3; u, v)) \right).$$

Thus, $(L_C\mathcal{M})_0$ looks like the nerve of the category whose objects are diagrams of the form

$$\begin{array}{ccccc} F_1(x_1) & \xrightarrow{u} & x_3 & \xleftarrow{v} & F_2(x_2) \\ \downarrow F_1(a_1) & & \downarrow a_3 & & \downarrow F_2(a_2) \\ F_1(x_1) & \xrightarrow{u} & x_3 & \xleftarrow{v} & F_2(x_2) \end{array}$$

where each $a_i \in \text{Aut}^h(x_i)$. In other words,

$$\text{Aut}^h((x_1, x_2, x_3; u, v))$$

consists of triples (a_1, a_2, a_3) such that the above diagram commutes.

For the moment, let us suppose that we have no homotopy invariance problems and that P_0 can be given by a pullback, rather than a homotopy pullback; further explanation on this point will be given shortly.

Since B is a right adjoint functor (see [12, III.1] for details), it commutes with pullbacks, and we have that

$$\begin{aligned} P_0 &\simeq \coprod_{\langle x_1 \rangle} B\text{Aut}^h(x_1) \times \coprod_{\langle x_3 \rangle} B\text{Aut}^h(x_3) \coprod_{\langle x_2 \rangle} B\text{Aut}^h(x_2) \\ &\simeq B \left(\coprod_{\langle x_1 \rangle} \text{Aut}^h(x_1) \times \coprod_{\langle x_3 \rangle} \text{Aut}^h(x_3) \coprod_{\langle x_2 \rangle} \text{Aut}^h(x_2) \right) \\ &\simeq B \left(\coprod_{\langle x_1, \langle x_2 \rangle, \langle x_3 \rangle} \text{Aut}^h(x_1) \times_{\text{Aut}^h(x_3)} \text{Aut}^h(x_2) \right). \end{aligned}$$

Thus, P_0 also looks like the nerve of the category whose objects are diagrams of the form given above, since the leftmost and rightmost vertical arrows are indexed by maps in $\text{Aut}(x_1)$ and $\text{Aut}(x_2)$, not by their images in \mathcal{M}_3 . So, if F_1 , for example, identifies two maps of $\text{Aut}(x_1)$, we still count two different diagrams. However, if we are taking a strict pullback, the horizontal maps must be equalities. We claim that taking the homotopy pullback, rather than the strict pullback, gives precisely all the diagrams as given above, without this restriction, as follows.

Since our diagram consists of fibrant objects in \mathcal{CSS} , we can apply [13, 19.9.4] and obtain a homotopy pullback by replacing one of the maps in the diagram with a fibration. In doing so, an object x_1 , for example, is replaced by a pair given by x_1 together with a map $F_1(x_1) \rightarrow x_3$. Doing the same for the other map (since there is no harm in replacing both of them by fibrations) we obtain all diagrams of the form given above. So, we have shown that we have the desired weak equivalence on level zero.

Now, it remains to show that we also get a weak equivalence of spaces at level one. The argument here is essentially the same but with larger diagrams. Again, we take ordinary pullbacks to reduce notation, but this issue can be resolved just as in the level zero case.

The space $P_1 = (L_C\mathcal{M}_1 \times_{L_C\mathcal{M}_3} L_C\mathcal{M}_2)_1$ can be written as follows:

$$\begin{aligned} &\left(\coprod_{\langle f_1: x_1 \rightarrow y_1 \rangle} B\text{Aut}^h(f_1) \right) \times \left(\coprod_{\langle f_3: x_3 \rightarrow y_3 \rangle} B\text{Aut}^h(f_3) \right) \left(\coprod_{\langle f_2: x_2 \rightarrow y_2 \rangle} B\text{Aut}^h(f_2) \right) \\ &\simeq B \left(\coprod_{\langle f_i: x_i \rightarrow y_i \rangle} (\text{Aut}^h(f_1) \times_{\text{Aut}^h(f_3)} \text{Aut}^h(f_2)) \right). \end{aligned}$$

Note that when we take $\langle f_i: x_i \rightarrow y_i \rangle$, the notation is meant to signify that we are varying x_i and y_i as objects, as well as maps between them, and then taking distinct weak equivalence classes.

On the other hand, if we let

$$f = (f_1, f_2, f_3): (x_1, x_2, x_3; u, v) \rightarrow (y_1, y_2, y_3; w, z),$$

the space $(L_C\mathcal{M})_1$ can be written as

$$\coprod_{\langle f \rangle} B\text{Aut}^h(f) \simeq B \left(\coprod_{\langle f \rangle} \text{Aut}^h(f) \right).$$

As above, let a_i denote a homotopy automorphism of x_i , and let b_i denote a homotopy automorphism of y_i . Then, both of the above spaces are given by the nerve of the category whose objects are diagrams of the form

$$\begin{array}{ccccc} F_1(x_1) & \xrightarrow{u} & x_3 & \xleftarrow{v} & F_2(x_2) \\ \downarrow F_1(f_1) & \searrow F_1(a_1) & \downarrow f_3 & \swarrow a_3 & \downarrow F_2(f_2) \\ & & F_1(x_1) & \xrightarrow{u} & x_3 & \xleftarrow{v} & F_2(x_2) \\ & & \downarrow F_1(f_1) & & \downarrow f_3 & & \downarrow F_2(f_2) \\ F_1(y_1) & \xrightarrow{w} & y_3 & \xleftarrow{z} & F_2(y_2) \\ \downarrow F_1(b_1) & \searrow F_1(b_1) & \downarrow b_3 & \swarrow b_3 & \downarrow F_2(b_2) \\ & & F_1(y_1) & \xrightarrow{w} & y_3 & \xleftarrow{z} & F_2(y_2) \end{array}$$

One could show that the higher-degree spaces of each of these complete Segal spaces are also weakly equivalent, but since these spaces are determined by these two, the above arguments are sufficient. \square

5. THE MORE GENERAL CONSTRUCTION ON COMPLETE SEGAL SPACES

In this section, we drop the condition that the maps u and v in the definition of the homotopy fiber product are weak equivalences in \mathcal{M}_3 and give a characterization of the resulting complete Segal space.

Again, let

$$\begin{array}{ccc} & & \mathcal{M}_2 \\ & & \downarrow F_2 \\ \mathcal{M}_1 & \xrightarrow{F_1} & \mathcal{M}_3 \end{array}$$

be a diagram of model categories and left Quillen functors. Let \mathcal{N} be the category whose objects are given by 5-tuples $(x_1, x_2, x_3; u, v)$, where x_i is an object of \mathcal{M}_i for each i , and the maps u and v fit into a diagram

$$F_1(x_1) \xrightarrow{u} x_3 \xleftarrow{v} F_2(x_2).$$

The 0-space of the complete Segal space $L_C\mathcal{N}$ has the homotopy type

$$\coprod_{\langle (x_1, x_2, x_3; u, v) \rangle} B\text{Aut}^h((x_1, x_2, x_3; u, v)).$$

An element of the group $\text{Aut}^h((x_1, x_2, x_3; u, v))$ looks like a diagram

$$\begin{array}{ccccc} F_1(x_1) & \xrightarrow{u} & x_3 & \xleftarrow{v} & F_2(x_2) \\ \simeq \downarrow & & \downarrow \simeq & & \downarrow \simeq \\ F_1(x_1) & \xrightarrow{u} & x_3 & \xleftarrow{v} & F_2(x_2). \end{array}$$

Using this diagram as a guide, we can formulate a concise description of $(L_C\mathcal{N})_0$.

Proposition 5.1. *Let N_0 denote the nerve of the category given by $(\cdot \rightarrow \cdot \leftarrow \cdot)$. The space $(L_C\mathcal{N})_0$ has the homotopy type of the pullback of the diagram*

$$\begin{array}{ccc} & & \text{Map}(N_0^t, L_C\mathcal{M}_3) \\ & & \downarrow \\ \text{Map}(\Delta[0]^t, L_C\mathcal{M}_1) \times \text{Map}(\Delta[0]^t, L_C\mathcal{M}_2) & \longrightarrow & \text{Map}(\Delta[0]^t, L_C\mathcal{M}_3)^2 \end{array}$$

where the horizontal map is given by

$$\text{Map}(\Delta[0]^t, L_C F_1) \times \text{Map}(\Delta[0]^t, L_C F_2)$$

and the vertical arrow is induced the pair of source maps $(s_1, s_2): N_0 \rightarrow \Delta[0] \amalg \Delta[0]$.

Proof. For $i = 1, 2, 3$, let a_i denote a homotopy automorphism of x_i in \mathcal{M}_i . The collection of diagrams of the form

$$\begin{array}{ccccc} F_1(x_1) & \xrightarrow{u} & x_3 & \xleftarrow{v} & F_2(x_2) \\ \downarrow F_1(a_1) & & \downarrow a_3 & & \downarrow F_2(a_2) \\ F_1(x_1) & \xrightarrow{u} & x_3 & \xleftarrow{v} & F_2(x_2). \end{array}$$

can be written as the pullback

$$\text{Aut}^h(u) \times_{\text{Aut}^h(x_3)} \text{Aut}^h(v).$$

Taking classifying spaces and coproducts over all isomorphism classes of objects, we obtain the pullback

$$(5.2) \quad \coprod_{\langle u: F_1(x_1) \rightarrow x_3 \rangle} \text{BAut}^h(u) \times \coprod_{\langle x_3 \rangle} \text{BAut}^h(x_3) \times \coprod_{\langle v: F_2(x_2) \rightarrow x_3 \rangle} \text{BAut}^h(v).$$

However, notice that the space

$$\coprod_{\langle u: F_1(x_1) \rightarrow x_3 \rangle} \text{BAut}^h(u)$$

is equivalent to the pullback

$$\coprod_{\langle x_1 \rangle} \text{BAut}^h(x_1) \times \coprod_{\langle x_3 \rangle} \text{BAut}^h(x_3) \times \coprod_{\langle f_3: y_3 \rightarrow x_3 \rangle} \text{BAut}^h(f_3),$$

since an element of $\text{Aut}^h(u)$ looks like a diagram

$$\begin{array}{ccc} y_3 & \xrightarrow{f_3} & x_3 \\ \simeq \downarrow & & \downarrow \simeq \\ y_3 & \xrightarrow{f_3} & x_3 \end{array}$$

where $y_3 = F_1(x_1)$.

Analogously, the space

$$\coprod_{\langle v: F_2(x_2) \rightarrow x_3 \rangle} \text{BAut}^h(v)$$

is equivalent to the pullback

$$\prod_{\langle x_2 \rangle} \text{BAut}^h(x_2) \times \prod_{\langle x_3 \rangle} \text{BAut}^h(x_3) \prod_{\langle f_3: y_3 \rightarrow x_3 \rangle} \text{BAut}^h(f_3).$$

Putting these two equivalences together, we get that the pullback (5.2) can be written as

$$\left(\prod_{\langle x_1 \rangle} \text{BAut}^h(x_1) \times \prod_{\langle x_3 \rangle} \text{BAut}^h(x_3) \prod_{\langle f_3 \rangle} \text{BAut}^h(f_3) \right) \times \prod_{\langle x_3 \rangle} \text{BAut}^h(x_3) \left(\prod_{\langle x_2 \rangle} \text{BAut}^h(x_2) \times \prod_{\langle x_3 \rangle} \text{BAut}^h(x_3) \prod_{\langle f_3 \rangle} \text{BAut}^h(f_3) \right).$$

However, this pullback can be written in a much more manageable way using our characterization of the complete Segal spaces corresponding to a model category. Thus, we get a pullback

$$(L_C \mathcal{M}_1)_0 \times_{(L_C \mathcal{M}_3)_0} (L_C \mathcal{M}_3^{[1]})_0 \times_{(L_C \mathcal{M}_3)_0} (L_C \mathcal{M}_2)_0 \times_{(L_C \mathcal{M}_2)_0} (L_C \mathcal{M}_3^{[1]})_0.$$

Rearranging terms in the pullback gives an equivalent formulation of this space as

$$((L_C \mathcal{M}_1)_0 \times (L_C \mathcal{M}_2)_0) \times_{(L_C \mathcal{M}_3)_0^2} ((L_C \mathcal{M}_3^{[1]})_0 \times_{(L_C \mathcal{M}_3)_0} (L_C \mathcal{M}_3^{[1]})_0).$$

However, this space is precisely the pullback of the diagram given in the statement of the proposition, since $\text{Map}(\Delta[0]^t, L_C \mathcal{M}_1) = (L_C \mathcal{M}_1)_0$ and analogously for \mathcal{M}_2 , and the pullback on the right agrees with the space $\text{Map}(N_0^t, L_C \mathcal{M}_3)$. \square

Now we give a characterization of the space $(L_C \mathcal{N})_1$.

Proposition 5.3. *If N_1 denotes the nerve of the category given by*

$$\begin{array}{ccc} \cdot & \xrightarrow{\quad} & \cdot & \xleftarrow{\quad} & \cdot \\ \downarrow & & \downarrow & & \downarrow \\ \cdot & \xrightarrow{\quad} & \cdot & \xleftarrow{\quad} & \cdot \end{array}$$

then the space $(L_C \mathcal{N})_1$ is weakly equivalent to the homotopy pullback of the diagram

$$\begin{array}{ccc} & & \text{Map}(N_1^t, L_C \mathcal{M}_3) \\ & & \downarrow \\ \text{Map}(\Delta[1]^t, L_C \mathcal{M}_1) \times \text{Map}(\Delta[1]^t, L_C \mathcal{M}_2) & \longrightarrow & \text{Map}(\Delta[1]^t, L_C \mathcal{M}_3)^2 \end{array}$$

where the maps are analogous to the ones in the previous proposition.

Proof. Again, let

$$f = (f_1, f_2, f_3): (x_1, x_2, x_3; u, v) \rightarrow (y_1, y_2, y_3; w, z).$$

Notice that, by definition, the homotopy type of the space $(L_C\mathcal{N})_1$ is given by

$$\coprod_{\langle f \rangle} B\text{Aut}^h(f).$$

An element of the group $\text{Aut}^h(f)$ is given by a diagram

$$\begin{array}{ccccc}
 F_1(x_1) & \xrightarrow{u} & x_3 & \xleftarrow{v} & F_2(x_2) \\
 \downarrow F_1(f_1) & \searrow F_1(a_1) & \downarrow f_3 & \searrow a_3 & \downarrow F_2(f_2) \\
 & & F_1(x_1) & \xrightarrow{u} & x_3 & \xleftarrow{v} & F_2(x_2) \\
 & & \downarrow F_1(f_1) & & \downarrow f_3 & & \downarrow F_2(f_2) \\
 F_1(y_1) & \xrightarrow{w} & y_3 & \xleftarrow{z} & F_2(y_2) \\
 \downarrow F_1(b_1) & \searrow F_1(b_1) & \downarrow F_1(f_1) & \searrow b_3 & \downarrow F_2(f_2) \\
 & & F_1(y_1) & \xrightarrow{w} & y_3 & \xleftarrow{z} & F_2(y_2)
 \end{array}$$

If we let $\alpha_1: u \rightarrow w$ and $\alpha_2: v \rightarrow z$ be maps in $\mathcal{M}^{[1]}$, such a diagram can also be regarded as an element of the homotopy pullback

$$\text{Aut}^h(f_1) \times_{\text{Aut}^h(f_3)} \text{Aut}^h(\alpha_1) \times_{\text{Aut}^h(f_3)} \text{Aut}^h(\alpha_2) \times_{\text{Aut}^h(f_2)} \text{Aut}^h(f_3).$$

Taking classifying spaces and coproducts over all possible classes of objects and morphisms, we obtain a pullback

$$\left(\coprod_{\langle f_1 \rangle} B\text{Aut}^h(f_1) \times \coprod_{\langle f_3 \rangle} B\text{Aut}^h(f_3) \coprod_{\langle \alpha_1 \rangle} B\text{Aut}^h(\alpha_1) \right) \\
 \times \coprod_{\langle f_3 \rangle} B\text{Aut}^h(f_3) \\
 \left(\coprod_{\langle f_2 \rangle} B\text{Aut}^h(f_2) \times \coprod_{\langle f_3 \rangle} B\text{Aut}^h(f_3) \coprod_{\langle \alpha_2 \rangle} B\text{Aut}^h(\alpha_2) \right).$$

This pullback can be rewritten in terms of the corresponding complete Segal spaces as

$$\left((L_C\mathcal{M}_3)_1 \times_{(L_C\mathcal{M}_3)_1} (L_C\mathcal{M}_3^{[1]})_1 \right) \times_{(L_C\mathcal{M}_3)_1} \left((L_C\mathcal{M}_2)_1 \times_{(L_C\mathcal{M}_3)_1} (L_C\mathcal{M}_3^{[1]})_1 \right).$$

At this point, notice that this space is also the pullback of the diagram

$$\begin{array}{ccc}
 & & (L_C\mathcal{M}_3^{[1]})_1 \times_{(L_C\mathcal{M}_3)_1} (L_C\mathcal{M}_3^{[1]})_1 \\
 & & \downarrow \\
 (L_C\mathcal{M}_1)_1 \times_{(L_C\mathcal{M}_3)_1} (L_C\mathcal{M}_2)_1 & \longrightarrow & (L_C\mathcal{M}_3)_1^2.
 \end{array}$$

However, since the upper space is equivalent to $\text{Map}(N_1^t, L_C\mathcal{M}_3)$, we have completed the proof. \square

Lastly, notice that the simplicial set N_1 from Proposition 5.3 is just $\text{Map}(\Delta[1], N_0) = N_0^{\Delta[1]}$, where N_0 is as in Proposition 5.1. We can use these results and the properties of complete Segal spaces to give the following theorem.

Theorem 5.4. *Let N be the simplicial space given by $N_n = N_0^{\Delta[n]}$. Then the complete Segal space $L_C \mathcal{N}$ is weakly equivalent to the homotopy pullback of the diagram*

$$\begin{array}{ccc} & \text{Map}(N, L_C \mathcal{M}_3) & \\ & \downarrow & \\ L_C \mathcal{M}_1 \times L_C \mathcal{M}_2 & \longrightarrow & L_C \mathcal{M}_3 \times L_C \mathcal{M}_3. \end{array}$$

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