

# DERIVED HALL ALGEBRAS FOR STABLE HOMOTOPY THEORIES

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ABSTRACT. In this paper we extend Toën’s derived Hall algebra construction, in which he obtains unital associative algebras from certain stable model categories, to one in which such algebras are obtained from more general stable homotopy theories, in particular stable complete Segal spaces satisfying appropriate finiteness assumptions.

## 1. INTRODUCTION

Hall algebras associated to abelian categories play an important role in representation theory. In particular, when the abelian category in question is the category of  $\mathbb{F}_q$ -representations of a quiver associated to a simply-laced Dynkin diagram, there is a close relationship between the Hall algebra and the quantum enveloping algebra of the Lie algebra associated to the same Dynkin diagram. Recent attempts to strengthen this relationship have led to the problem of associating some kind of Hall algebra to categories which are triangulated rather than abelian. In particular, it is conjectured that one could recover the quantum enveloping algebra from an appropriate Hall-type algebra associated to Peng and Xiao’s root category, which is, roughly speaking, the derived category of the category of this abelian category of representations, modulo a double shift relation [15].

In [22], Toën constructs “derived Hall algebras” associated to triangulated categories arising as homotopy categories of model categories whose objects are modules over a sufficiently finitary differential graded category over  $\mathbb{F}_q$ . In doing so, he develops a formula for the multiplication in this algebra in such a way that it can be regarded as a generalization of the formula for the multiplication in an ordinary Hall algebra. This formula was verified for more general triangulated categories, still satisfying certain finiteness conditions, by Xiao and Xu [24]. However, none of these methods can yet be applied to the root category, as it does not satisfy these finiteness assumptions.

In this paper, we seek to generalize Toën’s development of derived Hall algebras, using a modification of his proof to establish derived Hall algebras corresponding to triangulated categories arising as homotopy categories for more general stable

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homotopy theories. From the viewpoint of Xiao and Xu's work, we are not extending the class of triangulated categories for which derived Hall algebras can be defined; in fact, there may be triangulated categories for which we can obtain such an algebra using their methods, but which do not arise as such a homotopy category. However, most triangulated categories can be realized as homotopy categories, and, since a homotopy theory contains more information than its associated homotopy category, we seek to develop Toën's constructions in more generality so that we have a wider setting in which we can utilize this additional homotopical data. The hope is that such a construction will shed light on the question of how to find a similar algebra arising from a triangulated category which is not finitary.

In this paper, we use the complete Segal space model for homotopy theories. If we regard a homotopy theory as a category with weak equivalences, then there are several equivalent models for homotopy theories as mathematical objects, in particular objects of model categories with appropriate weak equivalences. Complete Segal spaces were developed by Rezk [17]; they are simplicial spaces satisfying conditions enabling one to regard them as something like a simplicial category up to homotopy. Their associated model category is in fact equivalent to the model structure on the category of simplicial categories [3], as well as to the model structures for Segal categories [3] and quasi-categories [10]. While any one of these models could be used, we prefer the complete Segal space model here because it is particularly well-suited for understanding fiber products of model categories [2], one of the key tools used by Toën in his proof of the associativity of derived Hall algebras. Specifically, we are able to use homotopy pullbacks of complete Segal spaces where he used the homotopy fiber product of model categories.

There is, in fact, another perspective on complete Segal spaces (and equivalent objects); they are also models for  $(\infty, 1)$ -categories, or  $\infty$ -categories with  $n$ -morphisms invertible for  $n > 1$ . While the motivation for using complete Segal spaces in this paper arises from the viewpoint that they are generalizations of model categories, it is also useful to remember that they can be thought of as generalizations of ordinary categories in this way.

## 2. STABLE MODEL CATEGORIES

Recall that a *model category*  $\mathcal{M}$  is a category with three distinguished classes of morphisms: weak equivalences, fibrations, and cofibrations, satisfying five axioms [5, 3.3]. Given a model category structure, one can pass to the *homotopy category*  $\mathrm{Ho}(\mathcal{M})$ , which is a localization of  $\mathcal{M}$  with respect to the class of weak equivalences [8, 1.2.1]. In particular, the weak equivalences, as the morphisms that we wish to invert, make up the most important part of a model category. An object  $x$  in a model category  $\mathcal{M}$  is *fibrant* if the unique map  $x \rightarrow *$  to the terminal object is a fibration. Dually, an object  $x$  in  $\mathcal{M}$  is *cofibrant* if the unique map  $\phi \rightarrow x$  from the initial object is a cofibration.

The standard notion of equivalence of model categories is given by the following definitions. First, recall that an *adjoint pair* of functors  $F: \mathcal{C} \rightleftarrows \mathcal{D}: G$  satisfies the property that, for any objects  $X$  of  $\mathcal{C}$  and  $Y$  of  $\mathcal{D}$ , there is a natural isomorphism

$$\varphi: \mathrm{Hom}_{\mathcal{D}}(FX, Y) \rightarrow \mathrm{Hom}_{\mathcal{C}}(X, GY).$$

The functor  $F$  is called the *left adjoint* and  $G$  the *right adjoint* [14, IV.1].

**Definition 2.1.** [8, 1.3.1] An adjoint pair of functors  $F: \mathcal{M} \rightleftarrows \mathcal{N}: G$  between model categories is a *Quillen pair* if  $F$  preserves cofibrations and  $G$  preserves fibrations. The left adjoint  $F$  is called a *left Quillen functor*, and the right adjoint  $G$  is called the *right Quillen functor*.

**Definition 2.2.** [8, 1.3.12] A Quillen pair of model categories is a *Quillen equivalence* if for all cofibrant  $X$  in  $\mathcal{M}$  and fibrant  $Y$  in  $\mathcal{N}$ , a map  $f: FX \rightarrow Y$  is a weak equivalence in  $\mathcal{D}$  if and only if the map  $\varphi f: X \rightarrow GY$  is a weak equivalence in  $\mathcal{M}$ .

We also consider model categories with the additional data that their homotopy categories are triangulated. Recall that a *triangulated category*  $T$  is an additive category, together with an equivalence  $\Sigma: T \rightarrow T$  called a *shift functor*, and a collection of *distinguished triangles*

$$x \xrightarrow{\alpha} y \xrightarrow{\beta} z \xrightarrow{\gamma} \Sigma x$$

satisfying four axioms [11, §2.1].

For a model category to have a triangulated homotopy category, it must first be *pointed*, in that its initial and terminal objects coincide. Such an object is called a *zero object*.

**Definition 2.3.** [8, 7.1.1] A pointed model category  $\mathcal{M}$  is *stable* if its homotopy category  $\text{Ho}(\mathcal{M})$  is triangulated.

**Example 2.4.** Let  $\mathcal{R}$  be a ring and  $Ch(\mathcal{R})$  the category of chain complexes of  $\mathcal{R}$ -modules. Then the model category structure on  $Ch(\mathcal{R})$  is triangulated. In fact, its homotopy category is equivalent to the *derived category*  $\mathcal{D}(\mathcal{R})$ , formed by taking  $Ch(\mathcal{R})$  modulo the equivalence relation given by chain homotopies of maps, and formally inverting the quasi-isomorphisms [11, §1.2].

### 3. STABLE COMPLETE SEGAL SPACES

**3.1. Simplicial spaces and complete Segal spaces.** Recall that the simplicial indexing category  $\Delta^{op}$  is defined to be the category with objects finite ordered sets  $[n] = \{0 \rightarrow 1 \rightarrow \dots \rightarrow n\}$  and morphisms the opposites of the order-preserving maps between them. A *simplicial set* is then a functor

$$K: \Delta^{op} \rightarrow \mathbf{Sets}.$$

We denote by  $\mathcal{SSets}$  the category of simplicial sets, and this category has a natural model category structure equivalent to the standard model structure on topological spaces [6, I.10].

One can consider more general simplicial objects; in this paper we work with *simplicial spaces* (also called bisimplicial sets), or functors

$$X: \Delta^{op} \rightarrow \mathcal{SSets}.$$

Given a simplicial set  $K$ , we also denote by  $K$  the simplicial space which has the simplicial set  $K$  at every level. We denote by  $K^t$ , or “ $K$ -transposed”, the constant simplicial space in the other direction, where  $(K^t)_n = K_n$ , where on the right-hand side  $K_n$  is regarded as a discrete simplicial set. The category of simplicial sets has a model category structure called the *Reedy structure* in which weak equivalences are given levelwise and all objects are cofibrant [16].

Specifically, we consider simplicial spaces satisfying additional conditions, namely, those inducing a notion of composition up to homotopy. These Segal spaces and

complete Segal spaces were first introduced by Rezk [17], and the name is meant to be suggestive of similar ideas first presented by Segal [21].

**Definition 3.1.** [17, 4.1] A *Segal space* is a Reedy fibrant simplicial space  $W$  such that the Segal maps

$$\varphi_n : W_n \rightarrow \underbrace{W_1 \times_{W_0} \cdots \times_{W_0} W_1}_n$$

are weak equivalences of simplicial sets for all  $n \geq 2$ .

Given a Segal space  $W$ , we can consider its *objects*  $\text{ob}(W) = W_{0,0}$ , and, between any two objects  $x$  and  $y$ , the *mapping space*  $\text{map}_W(x, y)$ , given by the homotopy fiber of the map  $W_1 \rightarrow W_0 \times W_0$  given by the two face maps  $W_1 \rightarrow W_0$ . The Segal condition stated above guarantees that a Segal space has a notion of  $n$ -fold composition of mapping spaces, up to homotopy.

The *homotopy category* of  $W$ , denoted  $\text{Ho}(W)$ , has as objects the elements of the set  $W_{0,0}$ , and

$$\text{Hom}_{\text{Ho}(W)}(x, y) = \pi_0 \text{map}_W(x, y).$$

A *homotopy equivalence* in  $W$  is a 0-simplex of  $W_1$  whose image in  $\text{Ho}(W)$  is an isomorphism. We consider the subspace of  $W_1$  whose components contain homotopy equivalences, denoted  $W_{\text{hoequiv}}$ . Notice that the degeneracy map  $s_0 : W_0 \rightarrow W_1$  factors through  $W_{\text{hoequiv}}$ ; hence we may make the following definition.

**Definition 3.2.** [17, §6] A *complete Segal space* is a Segal space  $W$  such that the map  $W_0 \rightarrow W_{\text{hoequiv}}$  is a weak equivalence of simplicial sets.

Given this definition, we can describe the complete Segal space model structure on the category of simplicial spaces.

**Theorem 3.3.** [17, 7.2] *There is a model structure CSS on the category of simplicial spaces such that the fibrant and cofibrant objects are precisely the complete Segal spaces. Furthermore, CSS has the additional structure of a cartesian closed model category.*

The fact that CSS is cartesian closed allows us to consider, for any complete Segal space  $W$  and simplicial space  $X$ , the complete Segal space  $W^X$ . In particular, using the simplicial structure, the simplicial set at level  $n$  is given by

$$(W^X)_n = \text{Map}(X \times \Delta[n]^t, W).$$

**3.2. Stable quasi-categories and complete Segal spaces.** As with model categories, we need to consider complete Segal spaces which are *stable*, in the sense that their homotopy categories are triangulated. It should be noted that, although we have given this simple definition of a stable complete Segal space, one could define it in a more technical way which permits a better understanding of the structure of a stable complete Segal space; Lurie has explained these ideas extensively for stable quasi-categories in [12], and they can fairly easily be translated into the equivalent setting of complete Segal spaces.

Although we do not go into this level of detail on this point in this paper, there are other notions that have been developed for quasi-categories which are useful here for complete Segal spaces. Thus, we give a very brief summary of quasi-categories and their relationship with complete Segal spaces.

Recall that a *quasi-category*  $X$  is a simplicial set satisfying the inner Kan condition, so that for any  $n \geq 1$  and  $0 < k < n$ , a dotted arrow lift exists in any diagram of the form

$$\begin{array}{ccc} V[n, k] & \longrightarrow & X \\ \downarrow & \nearrow \text{---} & \\ \Delta[n] & & \end{array}$$

The notion of quasi-category goes back to Boardman and Vogt [4], but it has received extensive attention more recently, especially by Joyal [9] and Lurie [13]. In particular, Joyal proves that there is a model category structure on the category of simplicial sets such that the fibrant and cofibrant objects are precisely the quasi-categories. We denote this category  $\mathcal{QCat}$ . Furthermore, Joyal and Tierney have proved that the model category  $\mathcal{QCat}$  is Quillen equivalent to Rezk's model category  $\mathcal{CSS}$  [10]. Remarkably, they prove that there are actually two different Quillen equivalences between these two model categories. Here, we make use of the one that is particularly easy to describe, the right adjoint  $\mathcal{CSS} \rightarrow \mathcal{QCat}$  given by  $W \mapsto W_{*,0}$ .

Using this relationship we return to the matter of explaining some necessary structures on complete Segal spaces. For a complete Segal space to be stable, we need it to be pointed, or to have a zero object, denoted  $0$ . As we have seen, in an ordinary category, a zero object is one which is both initial and terminal, so for any object  $x$ , there are unique morphisms  $x \rightarrow 0$  and  $0 \rightarrow x$ . As a complete Segal space is a homotopical generalization of a category, we require a homotopical notion of initial and terminal objects. The following definition, given by Joyal [9] and Lurie [13] for quasi-categories, is easy to reformulate for complete Segal spaces.

**Definition 3.4.** [13, 1.2.12.1, 1.2.12.6] An object  $x \in W_{0,0}$  of a complete Segal space is *initial* if it is initial as an object of  $\text{Ho}(W)$ , i.e., if  $\text{map}_W(x, y)$  is weakly contractible for any  $y \in W_{0,0}$ . Dually,  $x$  is *terminal* if it is terminal as an object of  $\text{Ho}(W)$ , i.e., if  $\text{map}_W(y, x)$  is weakly contractible for any  $y$ . An object is a *zero object* of  $W$  if it is both initial and terminal.

In addition to having a zero object, we need to have a notion of “pushout” within a complete Segal space, another analogue of a standard categorical idea within this generalized setting. Fortunately, formal definitions of limits and colimits within quasi-categories have been established by Lurie [13, 1.2.13.4]. We give a brief exposition here, enough to translate his definition into the world of complete Segal spaces; see [13, 1.2.8, 1.2.13] for a detailed exposition.

Let  $X$  and  $Y$  be simplicial sets. We can define their *join*  $X \star Y$  by

$$(X \star Y)_n = X_n \amalg Y_n \amalg \coprod_{i+j=n-1} X_i \times Y_j.$$

Note that the operation defines a monoidal product on  $\mathcal{SSets}$  with unit the empty simplicial set  $\phi$ . Then, for a fixed simplicial set  $X$ , we can define a functor

$$X \star (-): \mathcal{SSets} \rightarrow \mathcal{SSets}$$

by

$$Y \mapsto X \star Y$$

and notice that the map  $\phi \rightarrow Y$  is sent to the map  $X \star \phi = X \rightarrow X \star Y$ . Thus, the simplicial set  $X \star Y$  comes equipped with a canonical map  $X \rightarrow X \star Y$ , and so we

can regard  $X \star Y$  as an object of the *undercategory* or *category of simplicial sets under  $X$*  [14, II.6], denoted  $X \downarrow \mathcal{S}Sets$ . In doing so, we can think of our functor as

$$X \star (-): \mathcal{S}Sets \rightarrow X \downarrow \mathcal{S}Sets.$$

This functor has a right adjoint given by

$$(p: X \rightarrow Y) \mapsto Y.$$

To remember that  $Y$  has come from some map  $p: X \rightarrow Y$ , Lurie denotes the image of this functor  $Y_{p/}$ . We can think of  $Y_{p/}$  as the simplicial set  $Y$  with a specified  $X$ -shaped diagram inside it.

Such an object can be used to define colimits in a quasi-category. If  $Y$  is a quasi-category and  $p: X \rightarrow Y$  is a map of simplicial sets, then a *colimit* for  $p$  is an initial object of  $Y_{p/}$ . Dually, one could use the functor  $(-)\star X$ , its right adjoint, and the resulting definition of  $Y_{/p}$  to define a limit in a quasi-category  $Y$ .

Now, we translate this definition into  $\mathcal{CSS}$ .

**Definition 3.5.** Let  $W$  be a complete Segal space and  $X$  a simplicial set, together with a map  $p: X^t \rightarrow W$ . A *colimit* for  $p$  in  $W$  is an initial object of  $(W_{*,0})_{p/}$ , regarded as an object of  $W$ .

In this paper, we consider the case where the simplicial set  $X$  is  $\Delta[1] \amalg_{\Delta[0]} \Delta[1]$ , forming the diagram  $\cdot \leftarrow \cdot \rightarrow \cdot$ , so that the colimit is a “pushout” in the complete Segal space  $W$ . One can show that if  $W$  is stable, the fact that  $\mathrm{Ho}(W)$  is triangulated guarantees that colimits must always exist in  $W$ . Again, we refer the reader to Lurie’s manuscript on stable quasi-categories [12] for greater depth on this point.

**3.3. Model categories and complete Segal spaces.** We conclude this section with a brief exposition on the relationship between model categories and complete Segal spaces. Since we are translating a construction on model categories to one on complete Segal spaces, we need to understand how to regard a model category as a specific kind of complete Segal space.

As described by Rezk [17], any category with weak equivalences gives rise to a complete Segal space via the functor we denote  $L_{\mathcal{C}}$ ; given such a category  $\mathcal{C}$ ,  $L_{\mathcal{C}}\mathcal{C}$  is given by

$$(L_{\mathcal{C}}\mathcal{C})_n = \mathrm{nerve}(\mathrm{we}(\mathcal{C}^{[n]}))$$

where  $\mathrm{we}(\mathcal{C}^{[n]})$  denotes the category of weak equivalences of chains of  $n$  composable morphisms in  $\mathcal{C}$ .

If  $\mathcal{M}$  is a model category, then we can apply this construction, but, as explained in [2], it is only a functor when the morphisms between model categories preserve weak equivalences. Since we want a construction which is functorial on the category of model categories with left Quillen functors between them, we can modify the construction by restricting to the full subcategory of  $\mathcal{M}$  whose objects are cofibrant.

The main result of [1] is that this construction is well-behaved with respect to other natural ways of getting a complete Segal space from a model category; in particular, the resulting complete Segal space is weakly equivalent to the one obtained from taking the simplicial localization and then applying any one of several functors from simplicial categories to complete Segal spaces. There is an up-to-homotopy characterization of the resulting complete Segal space as well.

## 4. FIBER PRODUCTS OF MODEL CATEGORIES AND HOMOTOPY PULLBACKS OF COMPLETE SEGAL SPACES

A key tool in Toën's proof that his derived Hall algebras are associative is the fiber product of model categories. We begin with his definition as given in [22]. First, suppose that

$$\mathcal{M}_1 \xrightarrow{F_1} \mathcal{M}_3 \xleftarrow{F_2} \mathcal{M}_2$$

is a diagram of left Quillen functors of model categories. Define their *fiber product* to be the model category  $\mathcal{M} = \mathcal{M}_1 \times_{\mathcal{M}_3}^h \mathcal{M}_2$  whose objects are given by 5-tuples  $(x_1, x_2, x_3; u, v)$  such that each  $x_i$  is an object of  $\mathcal{M}_i$  fitting into a diagram

$$F_1(x_1) \xrightarrow{u} x_3 \xleftarrow{v} F_2(x_2).$$

A morphism of  $\mathcal{M}$ , say  $f: (x_1, x_2, x_3; u, v) \rightarrow (y_1, y_2, y_3; z, w)$ , is given by maps  $f_i: x_i \rightarrow y_i$  such that the following diagram commutes:

$$\begin{array}{ccc} F_1(x_1) & \xrightarrow{u} & x_3 & \xleftarrow{v} & F_2(x_2) \\ \downarrow F_1(f_1) & & \downarrow f_3 & & \downarrow F_2(f_2) \\ F_1(y_1) & \xrightarrow{z} & y_3 & \xleftarrow{w} & F_2(y_2). \end{array}$$

This category  $\mathcal{M}$  can be given the structure of a model category, where the weak equivalences and cofibrations are given levelwise. In other words,  $f$  is a weak equivalence (or cofibration) if each map  $f_i$  is a weak equivalence (or cofibration) in  $\mathcal{M}_i$ .

A more restricted definition of this construction requires that the maps  $u$  and  $v$  be weak equivalences in  $\mathcal{M}_3$ . Unfortunately, if we impose this additional condition, the resulting category cannot be given the structure of a model category because it does not have sufficient limits and colimits. However, it is still a perfectly good category with weak equivalences, and in some cases we can localize  $\mathcal{M}$  so that the fibrant-cofibrant objects of the localized model category have  $u$  and  $v$  weak equivalences [2]. Although Toën uses the model structure given above, at the point where he really makes use of the fiber product he restricts to the case where the maps  $u$  and  $v$  are weak equivalences. Thus, we assume here this extra structure.

Consider the functor  $L_C$ , described in the previous section, which takes a model category (or category with weak equivalences) to a complete Segal space. Given a fiber square of model categories where we require the maps  $u$  and  $v$  to be weak equivalences, we can apply this functor to obtain a commutative square

$$\begin{array}{ccc} L_C \mathcal{M} & \longrightarrow & L_C \mathcal{M}_2 \\ \downarrow & & \downarrow \\ L_C \mathcal{M}_1 & \longrightarrow & L_C \mathcal{M}_3. \end{array}$$

Alternatively, we could apply the functor  $L_C$  only to the original diagram and take the homotopy pullback, which we denote  $P$ , and obtain the following diagram:

$$\begin{array}{ccc} P & \longrightarrow & L_C \mathcal{M}_2 \\ \downarrow & & \downarrow \\ L_C \mathcal{M}_1 & \longrightarrow & L_C \mathcal{M}_3. \end{array}$$

**Theorem 4.1.** [2] *The complete Segal spaces  $L_C\mathcal{M}$  and  $P = L_C\mathcal{M}_1 \times_{L_C\mathcal{M}_3}^h L_C\mathcal{M}_2$  are weakly equivalent.*

This theorem allows us to use the homotopy pullback of complete Segal spaces to generalize the situations in which Toën uses the fiber product of model categories. In particular, we generalize a scenario given by Toën [22, 4.2] as follows.

Let

$$\begin{array}{ccc} W_0 & \xrightarrow{H_1} & W_1 \\ H_2 \downarrow & & \downarrow F_1 \\ W_2 & \xrightarrow{F_2} & W_3 \end{array}$$

be diagram of complete Segal spaces equipped with an isomorphism  $\alpha: F_1 \circ H_1 \Rightarrow F_2 \circ H_2$ , and define a map

$$F: W_0 \rightarrow W = W_1 \times_{W_3}^h W_2$$

by

$$x \mapsto (H_1(x), H_2(x); \alpha_x).$$

**Lemma 4.2.** *If  $\mathrm{Ho}(W_0) \rightarrow \mathrm{Ho}(W)$  is an equivalence of categories, then the diagram*

$$\begin{array}{ccc} \mathrm{nerve}(\mathrm{Ho}(wW_0)) & \longrightarrow & \mathrm{nerve}(\mathrm{Ho}(wW_1)) \\ \downarrow & & \downarrow \\ \mathrm{nerve}(\mathrm{Ho}(wW_2)) & \longrightarrow & \mathrm{nerve}(\mathrm{Ho}(wW_2)) \end{array}$$

*is homotopy cartesian.*

*Proof.* We want to show that the map

$$\mathrm{nerve}(\mathrm{Ho}(wW_0)) \rightarrow \mathrm{nerve}(\mathrm{Ho}(wW_1)) \times_{\mathrm{nerve}(\mathrm{Ho}(wW_3))}^h \mathrm{nerve}(\mathrm{Ho}(wW_2))$$

is a weak equivalence of simplicial sets. By our assumption, we know that the map

$$\mathrm{Ho}(W_0) \rightarrow \mathrm{Ho}(W_1 \times_{W_3}^h W_2)$$

is an equivalence of categories. Notice that, for each value of  $i$ , the homotopy category  $\mathrm{Ho}(wW_i)$  is the maximal subgroupoid of  $\mathrm{Ho}(W_i)$ . Hence, we have an equivalence of categories

$$\begin{aligned} \mathrm{Ho}(wW_0) &\rightarrow \mathrm{Ho}(w(W_1 \times_{W_3}^h W_2)) \simeq \mathrm{Ho}(wW_1 \times_{wW_3}^h wW_2) \\ &\simeq \mathrm{Ho}(wW_1) \times_{\mathrm{Ho}(wW_3)}^h \mathrm{Ho}(wW_2). \end{aligned}$$

Since nerves of equivalent categories are weakly equivalent simplicial sets, the lemma follows.  $\square$

## 5. HALL ALGEBRAS AND DERIVED HALL ALGEBRAS

**5.1. Classical Hall algebras.** Let  $\mathcal{A}$  be an abelian category. Throughout this section, we assume that  $\mathcal{A}$  is *finitary*, in that, for any objects  $x$  and  $y$  of  $\mathcal{A}$ , the groups  $\mathrm{Hom}(x, y)$  and  $\mathrm{Ext}^1(x, y)$  are finite.

**Definition 5.1.** [20] Given an abelian category  $\mathcal{A}$ , its *Hall algebra*  $\mathcal{H}(\mathcal{A})$  is defined as

- (1) the vector space with basis isomorphism classes of objects in  $\mathcal{A}$ , and

(2) multiplication given by

$$[x] \cdot [y] = \sum_{[z]} g_{x,y}^z [z]$$

where the *Hall numbers*  $g_{x,y}^z$  are given by

$$g_{x,y}^z = \frac{|\{0 \rightarrow x \rightarrow z \rightarrow y \rightarrow 0 \text{ exact}\}|}{|\text{Aut}(x)| \cdot |\text{Aut}(y)|}.$$

Notice that our assumptions on  $\mathcal{A}$  guarantee that each Hall number really is a finite number. It can be shown that this definition gives  $\mathcal{H}(\mathcal{A})$  the structure of a unital associative algebra [18].

Although Hall algebras have been investigated for a number of purposes, recent interest in them has arisen from the close relationship between Hall algebras and quantum groups in the following situation. Suppose that  $\mathfrak{g}$  is a Lie algebra of type  $A$ ,  $D$ , or  $E$ . Then  $\mathfrak{g}$  has an associated simply-laced Dynkin diagram, which is just an unoriented graph with no cycles. Assigning an orientation to each of the edges in this graph gives a quiver, or oriented graph, which we denote  $Q$ . Given a finite field  $\mathbb{F}_q$ , let  $\mathcal{A}$  be the category of  $\mathbb{F}_q$ -representations of this quiver  $Q$ . It can be shown that  $\mathcal{A}$  is in fact an abelian category satisfying our finiteness assumptions, and hence we have an associated Hall algebra  $\mathcal{H}(\mathcal{A})$  [18]. The Hall algebra as we have defined it is not independent of the chosen orientation on the quiver, but a slight modification by Ringel makes it so; this algebra is often called the *Ringel-Hall algebra* [19].

However, another algebra can be obtained from  $\mathfrak{g}$ , namely the quantum enveloping algebra  $U_q(\mathfrak{g})$ . This algebra can be given its triangular decomposition

$$U_q(\mathfrak{g}) = U_q(\mathfrak{n}^+) \otimes U_q(\mathfrak{h}) \otimes U_q(\mathfrak{n}^-).$$

Work of Ringel, further developed by Green, has shown that there is a close relationship between the Hall algebra  $\mathcal{H}(\mathcal{A})$  and the positive part of the quantum enveloping algebra,

$$U_q(\mathfrak{b}^+) = U_q(\mathfrak{n}^+) \otimes U_q(\mathfrak{h})$$

[7], [18].

A natural question to ask is whether there is some kind of enlarged version of the Hall algebra from which one could recover not just  $U_q(\mathfrak{b}^+)$ , but all of  $U_q(\mathfrak{g})$ . Work of Peng and Xiao [15] has led to the conjecture that such an algebra should be obtained from the following category. Using the abelian category  $\mathcal{A}$  of quiver representations as above, consider its bounded derived category  $\mathcal{D}^b(\mathcal{A})$ , which is no longer abelian, but is instead a triangulated category. As such, it has a shift functor  $\Sigma: \mathcal{D}^b(\mathcal{A}) \rightarrow \mathcal{D}^b(\mathcal{A})$ . We then define the *root category* of  $\mathcal{A}$  to be  $\mathcal{D}^b(\mathcal{A})/\Sigma^2$ , the triangulated category obtained from  $\mathcal{D}^b(\mathcal{A})$  by identifying an object with its double shift.

It is still an open question how to find a ‘‘Hall algebra’’ associated to this root category. For one thing, the usual definition does not apply because the root category is not abelian. It is, however, triangulated, and recent efforts in this area have focused on finding Hall algebras for triangulated categories. In the rest of this section, we describe derived Hall algebras, defined by Toën, which can be obtained from certain triangulated categories. Thus far the necessary restrictions on these triangulated categories prohibit us from being able to define a derived Hall algebra for the root category.

**5.2. Derived Hall algebras.** Recall that a *differential graded category* or *dg category*, is a category enriched over  $Ch(R)$ , the category of cochain complexes of modules over a ring  $R$ . Thus, given any objects  $x$  and  $y$  in a dg category  $\mathcal{T}$ , we have a cochain complex  $\mathcal{T}(x, y)$ . Here, we assume that  $R = \mathbb{F}_q$ , the finite field with  $q$  elements. Toën defines a dg category  $\mathcal{T}$  to be *locally finite* if for any objects  $x$  and  $y$  in  $\mathcal{T}$ , the cochain complex  $\mathcal{T}(x, y)$  is cohomologically bounded and has all cohomology groups finite dimensional [22, 3.1].

Given a locally finite dg category  $\mathcal{T}$ , we consider  $\mathcal{M}(\mathcal{T})$ , the category of dg  $\mathcal{T}^{op}$ -modules, or functors  $\mathcal{T} \rightarrow Ch(\mathbb{F}_q)$ . This category has the structure of a stable model category, with levelwise weak equivalences and fibrations [23, §3]. We have made finiteness assumptions about the dg category  $\mathcal{T}$ , but in taking the module category, we may have cochain complexes in the image which do not satisfy these kinds of conditions. If we restrict to functors which are appropriately finitary, we no longer have a model structure, since this subcategory does not possess enough limits and colimits. So, we work with the model category  $\mathcal{M}(\mathcal{T})$  of all modules but consider also the full subcategory  $\mathcal{P}(\mathcal{T})$  of perfect objects. A module in  $\mathcal{M}(\mathcal{T})$  is *perfect* if it belongs to the smallest subcategory of  $\text{Ho}(\mathcal{M}(\mathcal{T}))$  containing the quasi-representable modules (see [23, 3.6] for a definition) and which is stable by retracts, homotopy pushouts, and homotopy pullbacks [22]. Perfect objects coincide with the compact objects in the triangulated category  $\text{Ho}(\mathcal{M}(\mathcal{T}))$ . (Recall that if  $T$  is a triangulated category with arbitrary coproducts, then an object  $x$  of  $T$  is *compact* if any map  $x \rightarrow \coprod_i y_i$  factors through a finite coproduct [11, 6.5].)

Since  $\text{Ho}\mathcal{M}(\mathcal{T})$  is a triangulated category, it has a shift functor; we denote maps from  $x$  to the  $i$ th shift of  $y$  in this category by  $[x, y[i]]$  or by  $\text{Ext}^i(x, y)$ . Notice that for perfect modules, these Ext groups are all finite.

**Theorem 5.2.** [22, 1.1, 5.1] *Let  $\mathcal{T}$  be a locally finite dg category over a finite field  $\mathbb{F}_q$ . Define  $\mathcal{DH}(\mathcal{T})$  to be the  $\mathbb{Q}$ -vector space with basis the characteristic functions  $\chi_x$ , where  $x$  runs through the set of weak equivalence classes of perfect objects in  $\mathcal{M}(\mathcal{T})$ . Then there exists an associative and unital product*

$$\mu: \mathcal{DH}(\mathcal{T}) \otimes \mathcal{DH}(\mathcal{T}) \rightarrow \mathcal{DH}(\mathcal{T})$$

such that

$$\mu(\chi_x, \chi_y) = \sum_z g_{x,y}^z \chi_z$$

and these derived Hall numbers  $g_{x,y}^z$  are given by the formula

$$g_{x,y}^z = \frac{|[x, z]_y| \cdot \prod_{i>0} |\text{Ext}^{-i}(x, z)|^{(-1)^i}}{|Aut(x)| \cdot \prod_{i>0} |\text{Ext}^{-i}(x, x)|^{(-1)^i}},$$

where  $[x, z]_y$  denotes the subset of  $[x, z]$  of morphisms  $f: x \rightarrow z$  whose cone is isomorphic to  $y$  in  $\text{Ho}(\mathcal{M}(\mathcal{T}))$ .

## 6. MORE GENERAL DERIVED HALL ALGEBRAS

Throughout this section, suppose that  $W$  is a pointed stable complete Segal space, so that  $\text{Ho}(W)$  is a triangulated category with a zero object. As in the previous section, we define for any objects  $x, y$  in  $W$

$$\text{Ext}^i(x, y) = [x, y[i]]$$

where the outside brackets denote maps in  $\text{Ho}(W)$  and the inside brackets denote the shift functor giving the triangulated structure of  $\text{Ho}(W)$ .

**Definition 6.1.** A stable complete Segal space  $W$  is *finitary* if in  $\text{Ho}(W)$  we have that  $\text{Ext}^i(x, y)$  is finite for all pairs of objects  $(x, y)$  and all values of  $i$ , and zero for sufficiently large values of  $i$ .

We assume for the rest of the paper that all our stable complete Segal spaces are finitary.

Since the model category  $\mathcal{CSS}$  is cartesian closed, the simplicial space  $W^{\Delta[1]}$  is also a complete Segal space. Notice that  $W$  itself is isomorphic to the mapping object  $W^{\Delta[0]}$ , and so we can use the two maps  $\Delta[0] \rightarrow \Delta[1]$  to define “source” and “target” maps  $s, t: W^{\Delta[1]} \rightarrow W$ . Since an object of  $W^{\Delta[1]}$  is a 0-simplex  $u \in \text{map}_W(x, y)$  for some  $x$  and  $y$  objects of  $W$ , these two maps can be defined by  $s(u) = x$  and  $t(u) = y$ . We also have a “cone” map  $c: W^{\Delta[1]} \rightarrow W$  given by  $c(u) = y \amalg_x 0$ , where such a cone object exists because we have required that  $W$  be stable; in the homotopy category, it is just the completion of  $u: x \rightarrow y$  to a distinguished triangle.

Using these maps, we can put together the diagram

$$\begin{array}{ccc} W^{\Delta[1]} & \xrightarrow{t} & W \\ s \times c \downarrow & & \\ W \times W & & \end{array}$$

analogous to Toën’s diagram of model categories [22, §4].

Because we are no longer working with model categories, a number of aspects of this diagram have been simplified, compared to the analogous one in Toën’s paper. Because the objects are complete Segal spaces, rather than model categories, we no longer have to be concerned with whether these maps are left Quillen functors. Furthermore, we are able to impose conditions on  $W$  from the beginning so that its objects are already “perfect” in that all the necessary finiteness conditions are already satisfied.

A word on this point would perhaps be helpful here. It is likely that a stable complete Segal space that would arise in nature would not have all pairs of objects  $x$  and  $y$  satisfying the necessary finiteness conditions on  $\text{Ext}^i(x, y)$ . However, we can show that restricting to the sub-complete Segal space with objects satisfying such conditions is still a complete Segal space. Explicitly, given a complete Segal space  $W$ , consider the doubly constant simplicial space  $W_{0,0}$ , and the sub-simplicial space  $Z_{0,0}$  given by the perfect objects of  $W$ . Then define  $Z$  to be the simplicial spaces given by the pullback

$$\begin{array}{ccc} Z & \longrightarrow & W \\ \downarrow & & \downarrow \\ Z_{0,0} & \longrightarrow & W_{0,0}. \end{array}$$

Since  $W_{0,0}$  is discrete, the map  $W \rightarrow W_{0,0}$  is a fibration in  $\mathcal{CSS}$ , from which it follows that the map  $Z \rightarrow Z_{0,0}$  is a fibration also. Thus,  $Z$  is a fibrant simplicial space in  $\mathcal{CSS}$ , or a complete Segal space. Furthermore, since compact objects of

a triangulated category form a triangulated subcategory [11, 6.5],  $\mathrm{Ho}(Z)$  is triangulated and  $Z$  is stable. Thus, we can restrict to the appropriate setting without losing the structure that we need, and so we always assume that, given an arbitrary complete Segal space  $W$ , we have implicitly restricted to  $Z$ .

Now, as Toën does, we restrict to the sub-complete Segal spaces of  $W$  and  $W^{\Delta[1]}$ , whose mapping spaces are sent to isomorphisms in the homotopy category; we call these spaces  $wW$  and  $wW^{\Delta[1]}$ , respectively. Taking the nerve of the homotopy categories, we obtain a diagram

$$\begin{array}{ccc} \mathrm{nerve}(\mathrm{Ho}(wW^{\Delta[1]})) & \xrightarrow{t} & \mathrm{nerve}(\mathrm{Ho}(wW)) \\ s \times c \downarrow & & \\ \mathrm{nerve}(\mathrm{Ho}(wW)) \times \mathrm{nerve}(\mathrm{Ho}(wW)) & & \end{array}$$

For simplicity of notation, we write this diagram

$$\begin{array}{ccc} X^{(1)} & \xrightarrow{t} & X^{(0)} \\ s \times c \downarrow & & \\ X^{(0)} \times X^{(0)} & & \end{array}$$

To get an algebra with a well-defined multiplication, we need to show that this diagram of simplicial sets satisfies some properties.

**Definition 6.2.** [22, 2.1] An object  $X$  in the homotopy category of simplicial sets is *locally finite* if it satisfies the conditions

- (1) for any base point  $x \in X$  and  $i > 0$ , the group  $\pi_i(X, x)$  is finite, and
- (2) for any base point  $x \in X$ , there is some  $n$ , depending on  $x$ , such that  $\pi_i(X, x) = 0$  for all  $i > n$ .

**Lemma 6.3.** *The simplicial sets  $X^{(0)}$  and  $X^{(1)}$  are locally finite.*

*Proof.* For any  $x \in \pi_0(X^{(0)})$ , just as Toën does, we use the facts that

$$\pi_1(X^{(0)}) \subseteq \mathrm{Ext}^0(x, x) = [x, x]$$

and

$$\pi_i(X^{(0)}) = \mathrm{Ext}^{1-i}(x, x)$$

for  $i > 1$  [22, 3.2]. Our assumption on  $W$  guarantees that these groups are all finite, and that they are zero for sufficiently large  $i$ . Thus,  $X^{(0)}$  is locally finite.

To show that  $X^{(1)} = \mathrm{nerve}(\mathrm{Ho}(wW^{\Delta[1]}))$  is locally finite, notice that this space is weakly equivalent to

$$\mathrm{nerve}(\mathrm{Ho}(wW)) \times \Delta[1] = X^{(0)} \times \Delta[1]$$

which is also locally finite.  $\square$

**Definition 6.4.** [22, 2.5] A morphism  $f: X \rightarrow Y$  of locally finite homotopy types is *proper* if, for any  $y \in \pi_0(Y)$ , there are only finitely many  $x \in \pi_0(X)$  with  $f(x) = y$ .

Notice that  $f$  is proper if and only if, for any  $y \in \pi_0(Y)$ , the set  $\pi_0(F_y)$  is finite. The proof of the following lemma follows just as it does in Toën's paper [22, 3.2].

**Lemma 6.5.** *The map  $s \times c$  is proper.*

With these properties established for our diagram, we can use it to define an algebra much as Toën does [22, §4].

**Definition 6.6.** [22, 2.2] Let  $X$  be a simplicial set. The  $\mathbb{Q}$ -vector space of *rational functions with finite support* on  $X$  is the  $\mathbb{Q}$ -vector space of functions on the set  $\pi_0(X)$  with values in  $\mathbb{Q}$  and finite support, and is denoted by  $\mathbb{Q}_c(X)$ .

**Definition 6.7.** As a vector space, the *derived Hall algebra*  $\mathcal{DH}(W)$  of  $W$  is given by  $\mathbb{Q}_c(X^{(0)})$ .

Given a morphism  $f: X \rightarrow Y$  of locally finite simplicial sets, we define a push-forward morphism  $f_!: \mathbb{Q}_c(X) \rightarrow \mathbb{Q}_c(Y)$  as follows. Given  $y \in \pi_0(Y)$ , let  $F_y$  denote the homotopy fiber of  $f$  over  $y$ , and let  $i: F_y \rightarrow X$  be the natural map. Using the long exact sequences of homotopy groups, one can see that for any  $z \in \pi_0(F_y)$ , the group  $\pi_i(F_y, z)$  is finite for all  $i > 0$  and zero for sufficiently large  $i$ . Furthermore, the fibers of the map  $\pi_0(F_y) \rightarrow \pi_0(X)$  are all finite. Then, for any  $\alpha \in \mathbb{Q}_c(X)$  and  $y \in \pi_0(Y)$ , define the function  $f_!$  by

$$f_!(\alpha)(y) = \sum_{z \in \pi_0(F_y)} \alpha(i(z)) \cdot \prod_{i > 0} |\pi_i(F_y, z)|^{(-1)^i}.$$

The assumption that  $\alpha$  have finite support guarantees that  $f_!$  is well-defined.

If  $f: X \rightarrow Y$  is a proper map of locally finite homotopy types, then we have a well-defined pullback  $f^*: \mathbb{Q}_c(Y) \rightarrow \mathbb{Q}_c(X)$  defined in the usual way as  $f^*(\alpha)(x) = \alpha(f(x))$  for any  $\alpha \in \mathbb{Q}_c(Y)$  and  $x \in \pi_0(X)$ . The requirement that  $f$  be proper guarantees that  $f^*(\alpha)$  has finite support, so that  $f^*$  is in fact well-defined.

**Lemma 6.8.** [22, 2.6] *Consider a homotopy pullback diagram of locally finite homotopy types*

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ g \downarrow & & \downarrow f \\ Y' & \xrightarrow{u} & Y \end{array}$$

with  $u$  proper. Then the map  $v$  is also proper, and

$$u^* \circ f_! = g_! \circ v^*: \mathbb{Q}_c(X) \rightarrow \mathbb{Q}_c(Y').$$

To define the multiplication on  $\mathcal{DH}(W)$ , first notice that we have an isomorphism

$$\mathcal{DH}(W) \otimes \mathcal{DH}(W) \rightarrow \mathbb{Q}_c(X^{(0)} \times X^{(0)})$$

given by

$$(f, g) \mapsto ((x, y) \mapsto f(x) \cdot g(y)).$$

Then we can consider the map

$$\mu = t_! \circ (s \times c)^*: \mathcal{DH}(W) \otimes \mathcal{DH}(W) \rightarrow \mathcal{DH}(W)$$

The algebra structure on  $\mathcal{DH}(W)$  is then given by

$$x \cdot y = \sum_z g_{x,y}^z z$$

where

$$g_{x,y}^z = \mu(\chi_x, \chi_y)(z)$$

where  $\chi_x$  denotes the characteristic function of  $x$ .

**Proposition 6.9.** *With this multiplication,  $\mathcal{DH}(W)$  is a unital algebra.*

Our proof essentially follows the one given by Toën [22, 4.1], with the necessary changes being made as we translate to the complete Segal space setting.

*Proof.* Given any object  $x$  in  $W$ , let  $\chi_x$  denote its characteristic function; in particular, consider  $\chi_0$ , the characteristic function of the zero object of  $W$ .

Notice that the set  $\pi_0(X^{(1)})$  is isomorphic to the set of isomorphism classes of objects in  $\mathrm{Ho}(wW^{\Delta[1]})$ . Thus, fix some 0-simplex  $u: x \rightarrow y$  of  $\mathrm{map}_W(x, y)$ , regarded as an object of  $\mathrm{Ho}(wW^{\Delta[1]})$ . Then

$$(s \times c)^*(u) = \begin{cases} 1 & \text{if } y \cong 0 \text{ and } x \cong z \text{ in } \mathrm{Ho}(wW) \\ 0 & \text{otherwise.} \end{cases}$$

In other words,  $(s \times c)^*(\chi_0, \chi_x)$  is the characteristic function of the subset of  $\pi_0(X^{(1)})$  consisting of maps  $0 \rightarrow z$  with  $z \cong x$  in  $\mathrm{Ho}(wW)$ .

Define  $X$  to be the simplicial set contained in  $X^{(1)}$  consisting of all the support of  $(s \times c)^*(\chi_0, \chi_x)$ , and notice that  $X$  is a connected simplicial set [why is this?]. Then using the definition of the product map  $\mu$ , we get

$$\mu(\chi_0, \chi_x)(x) = \prod_{i>0} \left( |\pi_i(X)|^{(-1)^i} \cdot |\pi_i(X^{(0)}, x)|^{(-1)^{i+1}} \right).$$

Notice in particular that whenever  $y \neq x$ ,

$$\mu(\chi_0, \chi_x)(y) = 0.$$

Restricting the target map  $t: W^{\Delta[1]} \rightarrow W$  to the maps  $y \rightarrow z$  such that  $y \cong 0$  in  $\mathrm{Ho}(wW)$ , we see that on such objects  $t$  is fully faithful, up to homotopy. Thus, the induced map  $t: X \rightarrow X^{(0)}$  induces isomorphisms  $t_*: \pi_i(X) \rightarrow \pi_i(X^{(0)})$  for all  $i > 0$ , and the simplicial set  $X$  can be identified with a connected component of  $X^{(0)}$ . Hence,  $\mu(\chi_0, \chi_x)(x) = 1$ , so that  $\mu(\chi_0, \chi_x) = \chi_x$ .

Changing the order and following the same argument, one can see that we also have  $\mu(\chi_x, \chi_0) = \chi_x$ , thus proving that  $\chi_0$  is a unit element for  $\mathcal{DH}(W)$ .  $\square$

**Theorem 6.10.** *With this multiplication,  $\mathcal{DH}(W)$  is an associative algebra.*

*Proof.* Consider the complete Segal space  $W^{\Delta[2]}$ , and, as with  $W^{\Delta[1]}$  and  $W$ , denote by  $X^{(2)}$  the simplicial set  $\mathrm{nerve}(\mathrm{Ho}(wW^{\Delta[2]}))$ . Notice that there are three natural maps  $f, g, h: W^{\Delta[2]} \rightarrow W^{\Delta[1]}$  induced by the three inclusion maps  $\Delta[1] \rightarrow \Delta[2]$ , where  $f$  sends  $x \rightarrow y \rightarrow z$  to  $x \rightarrow y$ ,  $g$  sends it to  $y \rightarrow z$ , and  $h$  sends it to  $x \rightarrow z$ . There is also a cone map  $k: W^{\Delta[2]} \rightarrow W^{\Delta[1]}$  given by  $(x \rightarrow y \rightarrow z) \mapsto (y \amalg_x 0 \rightarrow z \amalg_x 0)$ , with the pushouts defined as before in a stable complete Segal space, and a map between the two given by the universal property. This map may not be unique, but all such maps form a weakly contractible space.

Using these maps, we get two diagrams:

$$\begin{array}{ccccc}
X^{(2)} & \xrightarrow{g} & X^{(1)} & \xrightarrow{t} & X^{(0)} \\
f \times (c \circ k) \downarrow & & \downarrow s \times c & & \\
X^{(1)} \times X^{(0)} & \xrightarrow{t \times \text{id}} & X^{(0)} \times X^{(0)} & & \\
(s \times c) \times \text{id} \downarrow & & & & \\
(X^{(0)} \times X^{(0)}) \times X^{(0)} & & & & 
\end{array}$$

and

$$\begin{array}{ccccc}
X^{(2)} & \xrightarrow{h} & X^{(1)} & \xrightarrow{t} & X^{(0)} \\
(s \circ f) \times k \downarrow & & \downarrow s \times c & & \\
X^{(0)} \times X^{(1)} & \xrightarrow{\text{id} \times t} & X^{(0)} \times X^{(0)} & & \\
\text{id} \times (s \times c) \downarrow & & & & \\
X^{(0)} \times (X^{(0)} \times X^{(0)}) & & & & 
\end{array}$$

which both give the same result taking composites across the top and down the left side:

$$\begin{array}{ccc}
X^{(2)} & \xrightarrow{\quad} & X^{(0)} \\
\downarrow & & \\
X^{(0)} \times X^{(0)} \times X^{(0)} & & 
\end{array}$$

Thus, to prove associativity of  $\mathcal{DH}(W)$ , it suffices by Lemma 6.8 to prove that the square in each of these diagrams is homotopy cartesian. In fact, it suffices to show that the diagrams

$$\begin{array}{ccc}
X^{(2)} \xrightarrow{g} X^{(1)} & & X^{(2)} \xrightarrow{h} X^{(1)} \\
f \downarrow & & \downarrow k \\
X^{(1)} \xrightarrow{t} X^{(0)} & & X^{(1)} \xrightarrow{t} X^{(0)} \\
\downarrow s & & \downarrow c
\end{array}$$

are homotopy cartesian. For the first diagram, this fact follows immediately from the fact that the original diagram

$$\begin{array}{ccc}
W^{\Delta[2]} \xrightarrow{g} W^{\Delta[1]} & & \\
f \downarrow & & \downarrow s \\
W^{\Delta[1]} \xrightarrow{t} W & & 
\end{array}$$

is a homotopy pullback diagram of complete Segal spaces. To show that the second diagram is homotopy cartesian requires more effort.

In this second diagram, let  $Z$  denote the homotopy pullback  $W^{\Delta[1]} \times_W^h W^{\Delta[1]}$ . Using Lemma 4.2, it suffices to prove that  $\text{Ho}(W^{\Delta[2]}) \rightarrow \text{Ho}(Z)$  is fully faithful and essentially surjective. We begin with the argument for the latter. Suppose we have an object  $(x \rightarrow z, w \rightarrow z \amalg_x 0)$  in  $\text{Ho}(Z)$ ; we want to find an object  $y$  of  $W$  such

that  $x \rightarrow y \rightarrow z$  is an object of  $\mathrm{Ho}(W^{\Delta[2]})$  with  $y \amalg_x 0 = w$ . Such a  $y$  can be found by applying the axioms for a triangulated category to the diagram

$$\begin{array}{ccccccc} x & \dashrightarrow & y & \dashrightarrow & w & \longrightarrow & x[1] \\ \downarrow = & & \downarrow & & \downarrow & & \downarrow = \\ x & \longrightarrow & z & \longrightarrow & z \amalg_x 0 & \longrightarrow & x[1]. \end{array}$$

To prove that the functor is fully faithful, we need to prove that, for any objects  $x \rightarrow y \rightarrow z$  and  $x' \rightarrow y' \rightarrow z'$  in  $\mathrm{Ho}(W^{\Delta[2]})$ , the map

$$\mathrm{Hom}_{\mathrm{Ho}(W^{\Delta[2]})}(x \rightarrow y \rightarrow z, x' \rightarrow y' \rightarrow z') \rightarrow \mathrm{Hom}_{\mathrm{Ho}(Z)}((x \rightarrow z, y \amalg_x 0 \rightarrow z \amalg_x 0), (x' \rightarrow z', y' \amalg_{x'} 0 \rightarrow z' \amalg_{x'} 0))$$

is an isomorphism. Elements of the set on the left-hand side are triples of maps making the diagram

$$\begin{array}{ccccc} x & \longrightarrow & y & \longrightarrow & z \\ \downarrow & & \downarrow & & \downarrow \\ x' & \longrightarrow & y' & \longrightarrow & z' \end{array}$$

commute, where elements of the set on the right-hand side are 4-tuples of maps making the pair of diagrams

$$\begin{array}{ccc} x & \longrightarrow & z \\ \downarrow & & \downarrow \\ x' & \longrightarrow & z', \end{array} \quad \begin{array}{ccc} y \amalg_x 0 & \longrightarrow & z \amalg_x 0 \\ \downarrow & & \downarrow \\ y' \amalg_{x'} 0 & \longrightarrow & z' \amalg_{x'} 0 \end{array}$$

commute. Given an element of the right-hand set, we can use the axioms for a triangulated category to find a map  $y \rightarrow y'$  compatible with the maps  $x \rightarrow x'$  and  $z \rightarrow z'$  to obtain an element of the left-hand set. Thus, the map is surjective. A similar argument can be used to prove that it is injective.  $\square$

The proof of the following formula is essentially the same as the one given by Toën [22, 5.1]; we give it here with the necessary changes to our situation.

**Proposition 6.11.** *The derived Hall numbers are given by*

$$g_{x,y}^z = \frac{|[x, z]_y| \cdot \prod_{i>0} |\mathrm{Ext}^{-i}(x, z)|^{(-1)^i}}{|\mathrm{Aut}(x)| \cdot \prod_{i>0} |\mathrm{Ext}^{-i}(x, x)|^{(-1)^i}},$$

where  $[x, z]_y$  denotes the subset of  $[x, z]$  of morphisms  $f: x \rightarrow z$  whose cone is isomorphic to  $y$  in  $\mathrm{Ho}(W)$ .

*Proof.* Given the target map  $t: X^{(1)} \rightarrow X^{(0)}$  and an object  $z$  of  $\mathrm{Ho}(W)$ , let  $F^z$  denote the homotopy fiber of  $t$  over  $z$ . Using the definitions of  $X^{(1)}$  and  $X^{(0)}$ , notice that  $F^z$  is weakly equivalent to the nerve of the category  $\mathrm{equiv}(W \downarrow z)$  whose objects are maps from arbitrary objects of  $W$  to  $z$ , and whose morphisms are the homotopy equivalences of  $W$ , making the resulting triangular diagram commute.

Given two other objects  $x$  and  $y$  of  $W$ , let  $F_{x,y}^z$  denote the nerve of the full subcategory of  $\mathrm{equiv}(W \downarrow z)$  whose objects are the maps  $u: x' \rightarrow z$ , where  $x' \simeq x$ , and whose cofiber is equivalent to  $y$ . Notice that  $F_{x,y}^z$  is locally finite, since both  $X^{(1)}$  and  $X^{(0)}$  are; moreover,  $\pi_0(F_{x,y}^z)$  is finite, and it is isomorphic to  $[x, z]_y/\mathrm{Aut}(x)$ .

Using  $F_{x,y}^z$ , we can reformulate our definition of the derived Hall number  $g_{x,y}^z$  as

$$g_{x,y}^z = \sum_{(u: x' \rightarrow y) \in \pi_0(F_{x,y}^z)} \prod_{i>0} |\pi_i(F_{x,y}^z, u)|^{(-1)^i}.$$

We first prove that

$$\prod_{i>0} |\pi_i(F_{x,y}^z, u)|^{(-1)^i} = |\text{Aut}(f/z)|^{-1} \cdot \prod_{i>0} |\text{Ext}^{-i}(x, z)|^{(-1)^i} \cdot |\text{Ext}^{-i}(x, x)|^{(-1)^{i+1}},$$

where  $\text{Aut}(f/z)$  denotes the stabilizer of a map  $f \in [x, z]_y$  under the action of  $\text{Aut}(x)$ .

Notice that we get a homotopy cartesian square of mapping spaces

$$\begin{array}{ccc} \text{map}_{W \downarrow z}(x, x) & \longrightarrow & \text{map}_W(x, x) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \text{map}_W(x, z) \end{array}$$

where the bottom horizontal map specifies the map  $u: x \rightarrow z$ . Thus, we have a fibration of simplicial sets, and hence a long exact sequence of homotopy groups

$$\begin{aligned} \cdots \rightarrow \pi_2(\text{map}(x, z)) \rightarrow \pi_1(\text{map}_{W \downarrow z}(x, x)) \rightarrow \pi_1(\text{map}_W(x, x)) \rightarrow \pi_1(\text{map}_W(x, z)) \\ \rightarrow \pi_0(\text{map}_{W \downarrow z}(x, x)) \rightarrow \pi_0(\text{map}_W(x, x)) \rightarrow \pi_0(\text{map}_W(x, z)) \rightarrow 0. \end{aligned}$$

Composing the last two maps between nontrivial sets, we get a surjection

$$\pi_0(\text{map}_{W \downarrow z}(x, x)) \rightarrow \text{Aut}(f/z).$$

Furthermore, notice that  $\pi_i(\text{map}_W(x, z)) = [x, z[-i]] = \text{Ext}^{-i}(x, z)$  and, similarly, that  $\pi_i(\text{map}_W(x, x)) = \text{Ext}^{-i}(x, x)$ . Finally, observe that  $\pi_i(\text{map}_{W \downarrow z}(x, x))$  is weakly equivalent to  $\pi_{i+1}(\text{nerve}(\text{equiv}(W \downarrow z)), u)$ , which, as we have noted previously, is equivalent to  $\pi_{i+1}(F_{x,y}^z, u)$ . Thus, we have a long exact sequence

$$\begin{aligned} \cdots \rightarrow \text{Ext}^{-2}(x, z) \rightarrow \pi_2(F_{x,y}^z, u) \rightarrow \text{Ext}^{-1}(x, x) \rightarrow \text{Ext}^{-1}(x, z) \\ \rightarrow \pi_1(F_{x,y}^z, u) \rightarrow \text{Aut}(f/z) \rightarrow 0. \end{aligned}$$

Using properties of long exact sequences, we obtain the equation given above.

To prove the statement of the proposition, we use the fact that, since  $\text{Aut}(x)$  is a finite group and  $[x, z]_y$  is a finite set, we get that

$$\frac{|[x, z]_y|}{|\text{Aut}(x)|} = \sum_{x \in ([x, z]_y / \text{Aut}(x))} |\text{Aut}(f/x)|.$$

The formula follows.  $\square$

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