Groupoids and Egyptian Fractions

Julie Bergner

University of California, Riverside

June 2, 2011

Groups

We begin with a review of groups.

Definition

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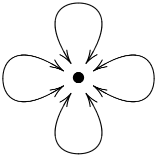
Example

1. $\mathbb{Z}$ under addition
2. $\mathbb{Z}/m$ under addition
3. S_n, the symmetric group on n letters (elements are the ways of permuting $\{1, 2, \ldots, n\}$)

A way to visualize groups

One way to think of a group is given by the following picture:



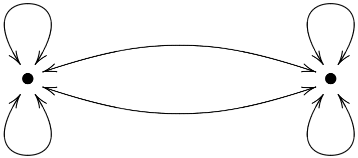
Group elements are given as arrows, with the reverse arrow being the inverse element.

We can think of them as "functions" from the center dot to itself.

All these "functions" start and end at the same place, so we can compose any of them in any order.

From groups to groupoids

We can draw more general pictures like this, but with more dots:



We can think of the set of arrows as having an operation which is defined whenever the arrows match up, so that the range of one is the domain of the other, and it is associative.

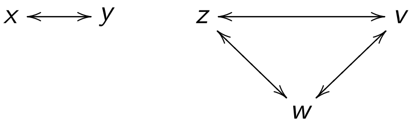
All arrows are required to have inverses, and each dot has an identity arrow.

Such a structure is called a *groupoid*.

Examples of groupoids

Example

Any equivalence relation on a set gives a groupoid. The elements of the set are the “dots” and there is an arrow between two elements if they are equivalent.



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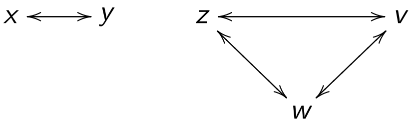
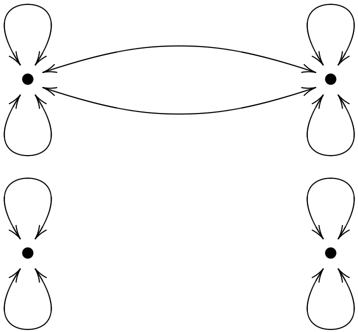


Example

The disjoint union $G \amalg H$ of two groups G and H is a groupoid.

Groups within a groupoid

Given any "dot" • in a groupoid, its *automorphisms* (arrows to and from that dot) form a group, $Aut(\bullet)$.



In fact, for any two dots in a connected component of a groupoid, their automorphism groups are isomorphic.

We denote by $[\bullet]$ the "equivalence class" of •, given by its connected component.

The order of a group

A group G has an *order*, just given by counting its elements:

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Example

1. $\#\mathbb{Z}/m = m$
2. $\#S_n = n!$
3. $\mathbb{Z}$ has infinite order

Similarly, we could count the arrows in a groupoid, but it doesn't turn out to be as useful.

Groupoid cardinality

Definition (Baez-Dolan)

The *groupoid cardinality* of a groupoid G is

$$|G| = \sum_{[\bullet]\in G} \frac{1}{\#Aut(\bullet)}.$$

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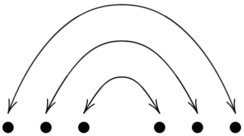
Example

1. If G is a group, then
$$|G| = \frac{1}{\# G}.$$
2. If G and H are groups, then
$$|G \amalg H| = \frac{1}{\# G} + \frac{1}{\# H}.$$

The following examples show why groupoid cardinality is a useful idea.

Example

Let G be the following groupoid:



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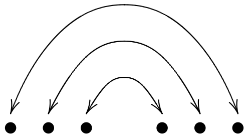
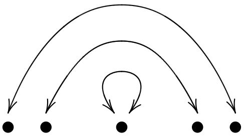


Then $|G| = 3$.

Example

In contrast, let H be the following groupoid:



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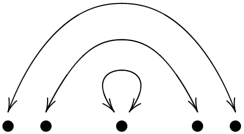

Then

$$|H| = \frac{5}{2}.$$

What kinds of numbers can we get?

Example

Let E be the groupoid with "dots" the finite sets and the "arrows" the isomorphisms between them.

The connected components will correspond to natural numbers.

Then we have

$$|E| = \sum_{[\bullet] \in E} \frac{1}{\#Aut(\bullet)}$$

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Finding groupoids with a given cardinality

Can we get *any* positive real number?

We can get any positive unit fraction from groups:

$$|\mathbb{Z}/n| = \frac{1}{n}.$$

(We could in fact use any group of order n.)

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We can also get whole numbers from dots with no non-identity arrows:

$$\bullet \qquad \bullet \qquad \bullet$$

For any rational number

$$\frac{m}{n} = \underbrace{\frac{1}{n} + \cdots + \frac{1}{n}}_{m}$$

we could use the groupoid

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$$\frac{m}{n} = \sum_i \frac{1}{n_i}$$

with each n_i distinct.

Egyptian fractions

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They also never repeated summands, so they would never write

$$\frac{1}{4} + \frac{1}{4} + \frac{1}{4}$$

but rather

$$\frac{1}{2} + \frac{1}{4}.$$

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Suppose I have five muffins that I want to divide among eight people. Using the fact that

$$\frac{5}{8} = \frac{1}{2} + \frac{1}{8}$$

it would be much easier to give each person half a muffin, and then an eighth of a muffin, than to try to cut pieces of size $\frac{5}{8}$ each.

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For example, we have

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but also

$$\frac{3}{4} = \frac{1}{2} + \frac{1}{5} + \frac{1}{20}.$$

Existence of Egyptian fraction decompositions

Theorem (Existence)

Every positive rational number has an Egyptian fraction decomposition.

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We'll begin by proving the second theorem.

A useful fact

Notice that

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Example

$$\frac{3}{4} = \frac{1}{2} + \frac{1}{4}$$

But using the fact, we have

$$\begin{aligned} \frac{1}{4} &= \frac{1}{4}\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6}\right) \\ &= \frac{1}{8} + \frac{1}{12} + \frac{1}{24}. \end{aligned}$$

Therefore, we get

$$\frac{3}{4} = \frac{1}{2} + \frac{1}{8} + \frac{1}{12} + \frac{1}{24}.$$

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One more iteration gives

$$\frac{3}{4} = \frac{1}{2} + \frac{1}{8} + \frac{1}{12} + \frac{1}{48} + \frac{1}{72} + \frac{1}{144}.$$

Proof of Infinitude.

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Assuming that $d_n > d_i$ for all $1 \leq i < n$, we get a new Egyptian fraction decomposition

$$q = \frac{1}{d_1} + \ldots + \frac{1}{2d_n} + \frac{1}{3d_n} + \frac{1}{6d_n}.$$

□

The greedy algorithm

The following strategy, called the “greedy algorithm”, is due to Fibonacci.

Suppose that we have a rational number

$$\frac{a}{b}$$

with $a > 1$.

We find the largest unit fraction smaller than it, and repeat until we get an Egyptian fraction decomposition.

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Let's look at how this works for the fraction

$$\frac{5}{21}.$$

Example

We know that

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Repeating, we have

$$\frac{1}{27} = \frac{4}{108} < \frac{4}{105} < \frac{4}{104} = \frac{1}{26}$$

and subtracting we get

$$\frac{5}{21} = \frac{1}{5} + \frac{1}{27} + \frac{1}{945}.$$

Proof of Existence

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Then

$$\begin{aligned}\frac{a}{b} &= \frac{1}{d_1} + \left(\frac{a}{b} - \frac{1}{d_1}\right) \\ &= \frac{1}{d_1} + \frac{ad_1 - b}{bd_1}.\end{aligned}$$

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So, the numerator of our new fraction is smaller than the original numerator a.

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$$ad_1 - b > 1,$$

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But, we can repeat this process to find d_2 such that

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But, we can repeat this process to find d_2 such that

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Since the numerators are always positive integers and always strictly decreasing, at some point this process must give a numerator of 1, terminating the process. □

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Example

Applying the greedy algorithm to

$$\frac{83}{140}$$

yields

$$\frac{83}{140} = \frac{1}{2} + \frac{1}{11} + \frac{1}{514} + \frac{1}{395780}.$$

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Example

Applying the greedy algorithm to

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yields

$$\frac{83}{140} = \frac{1}{2} + \frac{1}{11} + \frac{1}{514} + \frac{1}{395780}.$$

However, this fraction can also be decomposed as

$$\frac{83}{140} = \frac{1}{4} + \frac{1}{5} + \frac{1}{7}.$$

Back to groupoid cardinality

We can now apply these theorems to our goal of finding a groupoid with a given cardinality.

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Theorem

Any positive rational number is a the groupoid cardinality of infinitely many distinct groupoids.

We can even get positive *real* numbers if we allow convergent series of unit fractions.

What is this good for?

Groupoid cardinality appears in recent work by Baez-Hoffnung-Walker on a process called "groupoidification".

It is a process of using groupoids to understand vector spaces with additional algebraic structure, for example Hecke algebras and Hall algebras.

Understanding different ways to get a groupoid with given cardinality could give different choices for "groupoidification".

Further questions

1. Is finding the shortest Egyptian fraction decomposition useful for groupoid applications?
2. Of the many methods of finding Egyptian fraction decompositions, are any preferable here?
3. Should we be considering groupoids other than disjoint unions of cyclic groups in these examples?
4. Are there other facts about Egyptian fractions that can be applied to groupoid cardinality?