Finding Rates and the Geometric Mean

So far, most of the situations we've covered have assumed a known interest rate. If you save a certain amount of money and it earns a fixed interest rate for a period of time, how much money will you have at the end? What we'd like to consider here is the case where you know how much money you've invested and how much you have earned and would like to know the percent increase (hoping that it has indeed increased). This situation might come up if you are investing in stocks, where the rate of return changes frequently, or if you have your money invested in a few different places and you want to know how well you have done overall.

Let's start with a fairly straightforward situation. Suppose you invested \$1000. After one year you have \$1075. If we assume that the interest was compounded annually, what was the interest rate? Since this problem just uses simple interest, we have that

$$1075 = 1000(1 + r).$$

Solving for r, we get

$$1 + r = \frac{1075}{1000} = 1.075.$$

Thus, the rate is $r = .075$ or 7.5%.

Now, what if we have a period of several years? Suppose you invest \$1000, and after 3 years you have 1500. If the interest is computed annually, what is the rate? Now we need to use our formula for compound interest:

$$1500 = 1000(1 + r)^3.$$

We start as we did previously, by dividing both sides by 1000 to get

$$(1 + r)^3 = 1.5.$$

Now we need to take the cube root of both sides to undo the exponent of 3 on the left side of the equation. (On the calculator, type "3" and then under "MATH" find $\sqrt[x]{}$, and then type the number you want to take the cube root of. There is a also a $\sqrt[3]{}$ under "MATH," but we will also need to do this process for other values of x.) Now we have

$$1 + r = \sqrt[3]{1.5},$$

or

$$r = \sqrt[3]{1.5} - 1 \approx .145.$$

Thus, the interest rate was about 14.5%, so a very good rate of return.

Now, suppose that you knew that your interest was being compounded monthly. If you invested \$100 and one year later you have \$125, what is the interest rate? Now, we use the formula

$$125 = 100\left(1 + \frac{r}{12}\right)^{12}.$$

Dividing both sides by 100 gives

$$\frac{5}{4} = \left(1 + \frac{r}{12}\right)^{12}.$$

Now, we need to take the 12th root of both sides. It looks scarier, but it is the exact same process:

$$1 + \frac{r}{12} = \sqrt[12]{\frac{5}{4}}.$$

Now, solving for r gives

$$\frac{r}{12} = \sqrt[12]{\frac{5}{4}} - 1$$

$$r = 12\left(\sqrt[12]{\frac{5}{4}} - 1\right) \approx .225.$$

Thus, the interest rate is about 22.5%.

Exercises:

1. Suppose you invested \$1000. Ten years later you had \$3000. If the interest was compounded annually, what was the rate?
2. Suppose you invested \$100. One year later you had \$110. If the interest was compounded semi-annually, what was the interest rate?
3. Suppose you invested \$2000. Five years later you had \$3000. If the interest was compounded monthly, what was the interest rate?

Now, suppose that you knew the rate of return each year and wanted to know the average rate of return. Let's begin by recalling a problem mentioned yesterday in class. Suppose you have \$100 in a stock whose value fluctuates frequently. In the first year, you earn 10%, so at the end of the year you have \$110. In the second year, you lose 10%. Thus, you lose \$11 and end with \$99. The average rate of return is definitely not 0, even though the percentages look like they should cancel each other out! The problem is, in the second year you have more to lose. But what is the average rate of return for the two years? Well, you start with \$100 and end with \$99. Thus, we have the formula

$$99 = 100(1 + r)^2.$$

(We'll just assume we compound annually.) Solving for r, we get

$$(1 + r)^2 = .99$$
$$1 + r = \sqrt{.99}$$
$$r = \sqrt{.99} - 1 \approx -.005.$$

Thus, our rate of return overall is $-.5\%$.

You might be wondering at this point if you would be better off if you had lost the 10% before you gained 10%. So, let's suppose that you start with \$100 and lose 10%. Then you lose \$10 and are left with \$90. Then, suppose in the second year you gain 10%. Well, 10% of 90 is 9, so we end up with \$99, just like we did before!

Why is this? Well, let's look at the formula. In the first case, to find the amount after one year we multiplied

$$100(1 + .10) = 100(1.10).$$

Then, to compute the second year, we multiplied this amount by $(1 + (-.10)) = (.90)$. Thus, we got

$$100(1.10)(.90).$$

In the second situation, we just reverse the order and get

$$100(.90)(1.10).$$

Since multiplication is commutative, it doesn't matter which one you take first.

Thus, when we compute average rates of return, it will not matter in which year we received which rate. Let's look at another example.

Suppose your rate of return for three different years is 10%, 3%, and 6%. What is your average rate of return? Let's suppose that you have invested \$2000.. By the process used in the previous example, we get

$$2000(1.10)(1.03)(1.06).$$

To find the average rate of return, we set this equal to

$$2000(1+r)^3$$

and solve for r:

$$2000(1.10)(1.03)(1.06) = 2000(1+r)^3$$

$$(1.10)(1.03)(1.06) = (1+r)^3$$

$$1.20098 = (1+r)^3$$

$$\sqrt[3]{1.20098} = 1+r$$

$$r = \sqrt[3]{1.20098} - 1 \approx .063.$$

Thus, the average rate was about 6.3%.

Notice in particular that the amount we invested didn't matter; the rate is independent it. Once we set up our equation, the first thing we did was divide both sides by 2000, or the amount invested.

Exercise 4: Suppose for four years your rates of return are 15%, −4%, −1%, and 3%. What is the average rate of return?

An important thing to notice here is that to take the "average" rate we are *not* adding up the interest rates and dividing by the number of years. Instead, we are multiplying the interest rates and taking the nth root, where n is the number of years. This kind of "average" is called the *geometric mean*, whereas the usual average formula is called the *arithmetic mean.*

To further make sense of this vocabulary, recall that on Tuesday we talked about *geometric* series. We started by discussing sequences in which each term was obtained from the previous one by adding a fixed number. For example, the sequence

$$0, 5, 10, 15, 20, \ldots$$

is obtained by adding 5 to each term to get the next one. This kind of sequence is natural even for young children, since it is just skip-counting. Such a sequence is called an *arithmetic sequence*. If you have a missing term, you can find it by taking the arithmetic mean. As an example, consider the sequence

$$1, 4, 7, 10, 13, n, 19, \ldots$$

If you see the pattern (adding 3), you can figure out the value of n. But, another way to do it is to average the numbers coming before and after it:

$$\frac{13 + 19}{2} = \frac{32}{2} = 16.$$

Then, if you add the terms of an arithmetic sequence, you get an *arithmetic series*:

$$1 + 4 + 7 + 10 + 13 + 16 + 19 + \cdots$$

We then went on to talk about sequences in which each term is obtained by multiplying the previous term by a fixed number. For example,

$$1, 3, 9, 27, 81, \ldots$$

This kind of sequence is called a *geometric sequence* and if we add the terms together, we get a *geometric series*:

$$1 + 3 + 9 + 27 + 81 + \cdots$$

If we had a geometric sequence and needed to know what a missing term was, we would use the geometric mean, rather than the arithmetic mean, to find it. For example,

$$1, 2, 4, 8, n, 32, \ldots$$

To find n, we take

$$\sqrt{8 \cdot 32} = \sqrt{256} = 16.$$

Notice that there is a progression of operations here. Iterated addition is multiplication, and then iterated multiplication is raising to a power. In the arithmetic mean, we add two (or more) numbers, and then divide (the "reverse" of multiplication) by the number of terms. In the geometric mean, we go up a level. We multiply the terms and then take the nth root (the "reverse" of raising to the nth power), where n is the number of terms.

We don't need an arithmetic or geometric sequence to take these means, however. In fact, we can compare the two. Consider the numbers $3, 9, 3, 1$. Their arithmetic mean is

$$\frac{3+9+3+1}{4} = \frac{16}{4} = 4.$$

Their geometric mean, on the other hand, is

$$\sqrt[4]{3 \cdot 9 \cdot 3 \cdot 1} = \sqrt[4]{81} = 3.$$

So, we don't necessarily get the same thing.

Exercise 5: What is the arithmetic mean of the numbers $1, 2, 3, 4$? What is their geometric mean?