Geometric Series and Annuities

Our goal here is to calculate annuities. For example, how much money do you need to have saved for retirement so that you can withdraw a fixed amount of money each year for 30 years? If the lottery promises to pay you 10 million dollars over 10 years, how much is that worth today?

We begin with the mathematical techniques we will need here. In particular, we need to consider *series*. An example of a series is

$$1 + 2 + 3 + 4 + 5 + \cdots .$$

We are repeatedly adding terms, and if we keep adding forever we call it an *infinite series*. To see what is happening with a series, we can start adding terms.

$$1 + 2 = 3$$

$$1 + 2 + 3 = 6$$

$$1 + 2 + 3 + 4 = 10$$

$$1 + 2 + 3 + 4 + 5 = 15$$

We could keep going. For this particular series, if we keep adding terms, the number we get is going to keep getting larger and go to ∞.

For the series we just considered, we got each new term by adding 1 to the previous term. The series we will be interested in today are those in which each term is obtained from the previous one by multiplying by a fixed number. For example, consider the series

$$1 + 2 + 4 + 8 + 16 + \cdots .$$

We get each new term by multiplying by 2. In fact, we could rewrite this series as

$$1 + 2 + 2^2 + 2^3 + 2^4 + \cdots + 2^n + \cdots .$$

This kind of series is called a *geometric series*. As with the previous series, as we keep adding terms, the result will get larger.

Is there an example where we don't go off to ∞? Consider the series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

and look at the first few sums:

$$1 + \frac{1}{2} = \frac{3}{2} = 1.5$$

$$1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4} = 1.75$$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8} = 1.875$$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{31}{16} = 1.9375$$

It looks like this series is "converging" to 2, in that the sums are getting closer to 2. How do we know for sure?

Note that this is a geometric series where we are multiplying by $\frac{1}{2}$. So, our series can be written instead as

$$1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \cdots + \left(\frac{1}{2}\right)^n + \cdots$$

A geometric series is always going to look like

$$1 + a + a^2 + a^3 + \cdots + a^n + \cdots$$

for some number a. We've seen the example where $a = 2$ (and the sums went to ∞) and the example where $a = \frac{1}{2}$, where the sums appeared to be converging to 2. What is the difference between the two?

There is a formula which says that if $-1 < a < 1$, then our geometric series will converge to the following:

$$1 + a + a^2 + a^3 + \cdots + a^n + \cdots = \frac{1}{1-a}.$$

Let's apply this formula to our example where $a = \frac{1}{2}$. Then we get

$$1 + a + a^2 + a^3 + \cdots + a^n + \cdots = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2.$$

So, our guess was correct.

Note that we cannot use this formula for the example where $a = 2$, since $2 > 1$.

However, what if we wanted to stop adding terms at some n instead of adding terms forever? Again, we have a formula:

$$1 + a + a^2 + a^3 + \cdots + a^n = \frac{1 - a^{n+1}}{1 - a}.$$

For example,

$$1 + a + a^2 + a^3 + \cdots + a^{10} = \frac{1 - (\frac{1}{2})^{11}}{1 - \frac{1}{2}} \approx 1.999.$$

If you are interested in some notation here, we can write

$$1 + a + a^2 + a^3 + \cdots + a^n = \sum_{i=0}^{n} a^i$$

and

$$1 + a + a^2 + a^3 + \cdots + a^n + \cdots = \sum_{i=0}^{\infty} a^i.$$

This is called summation notation and is a convenient way to write a series.

Now, what does this have to do with finance? Let's look at an example. Suppose you win the lottery, and you will be paid $\$10,000,000$ over 10 years, in other words, $\$1,000,000$ per year for 10 years. If the money is invested at 8% interest, compounded annually, how much money does the lottery have to have saved up in order to pay you your $\$1,000,000$ each year?

Suppose that you will be paid $\$1,000,000$ now and then again for 9 years after this. So, first of all, they need $\$1,000,000$ to pay you right now. However, since the money will be earning interest, they don't need to have a full million dollars in order to pay you the same amount next year. Recall from yesterday's class that if you invest an amount p_1 at 8% interest, at the end of the year you will have $p_1(1.08)$. In this problem, we know how much we want at the end and instead want to know what p_1 needs to be. So, we have

$$1,000,000 = p_1(1.08).$$

or, solving for p_1,

$$p_1 = \frac{1,000,000}{1.08}.$$

So, the lottery people had better have

$$p_1 = \frac{1,000,000}{1.08} \approx 925,925.93$$

saved in order to pay you one year from now.

What about two year from now? Now the money has two years to collect interest, so if we call p_2 the amount that needs to be saved in order to pay you 2 years from now, we have

$$1,000,000 = p_2(1.08)^2.$$

or

$$p_2 = \frac{1,000,000}{(1.08)^2}.$$

So, in order to pay for the nth year, there needs to be an additional

$$p_n = \frac{1,000,000}{(1.08)^n}.$$

Thus, the total the lottery people need to have now is

$$1,000,000 + \frac{1,000,000}{1.08} + \frac{1,000,000}{(1.08)^2} + \cdots + \frac{1,000,000}{(1.08)^9}.$$

Let's factor out the 1,000,000 to make the numbers more manageable (and get a geometric series):

$$1,000,000\left(1 + \frac{1}{1.08} + \frac{1}{(1.08)^2} + \cdots + \frac{1}{(1.08)^9}\right)$$

which we can rewrite as

$$1,000,000\left(1 + \frac{1}{1.08} + (\frac{1}{1.08})^2 + \cdots + (\frac{1}{1.08})^9\right).$$

Now we can apply our formula:

$$1,000,000\left(1 + \frac{1}{1.08} + (\frac{1}{1.08})^2 + \cdots + (\frac{1}{1.08})^9\right) = \frac{1 - (\frac{1}{1.08})^{10}}{1 - \frac{1}{1.08}} = 7,250,000.$$

Thus, in order to pay you your $\$10,000,000$ over 10 years, the lottery must have $\$7,250,000$ saved up now.

To make the formula more specific, if you have a payout amount of k dollars per year for n years (assuming the first year is paid out immediately) and the money is invested at an annual interest rate r, the money that needs to be invested initially (also called the *present value*) is given by

$$k\left(\frac{1-(\frac{1}{1+r})^n}{1-\frac{1}{1+r}}\right).$$

As another example, suppose that you are planning for retirement. You are able to invest your money at 10% interest (compounded annually) and you would like to withdraw $40,000 per year for 30 years. How much do you need to have saved? Applying the formula, we get

$$40,000\left(\frac{1-(\frac{1}{1.10})^{30}}{1-\frac{1}{1.10}}\right) \approx \$414,784.24.$$

When might you use the infinite series formula? An example here is an endowment, where you want a fixed amount paid indefinitely. For example, suppose you wanted to set up a scholarship fund to pay a worthy student $1000 each year, and you wanted this fund to keep going long into the future. Further suppose this money could be invested at 12% interest. Then you would use the infinite geometric series formula as above to get

$$1000\left(\frac{1}{1-\frac{1}{1.12}}\right) \approx 9333.33.$$

Thus, $9,333.33 would be enough money to pay for your scholarship fund as far into the future as you'd like.