MATH 246B ALGEBRAIC TOPOLOGY II, HOMEWORK EXERCISES

1. Due Monday, April 6

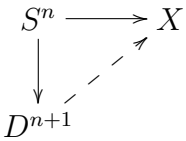
(1) Let $f\colon X \to Y$ be a homotopy equivalence. Show that for all $n \geq 0$ the map

$$f_*\colon \pi_n(X) \to \pi_n(Y)$$

is an isomorphism.

2. Due Monday, April 13

(1) Show that the following statements are equivalent for any $n \geq 0$:
   (a) $\pi_n(X) = 0$.
   (b) Every map $S^n \to X$ is homotopic to a constant map.
   (c) Every map $S^n \to X$ extends to a map $D^{n+1} \to X$, i.e., a lifting exists



   making the diagram commute.

(2) Show that the following statements are equivalent for any $n \geq 1$:
   (a) $\pi_n(X, A) = 0$.
   (b) Every map $(D^n, \partial D^n) \to (X, A)$ is homotopic rel ∂D^n to a map $D^n \to A$.
   (c) Every map $(D^n, \partial D^n) \to (X, A)$ is homotopic through maps of pairs to a map $D^n \to A$.
   (d) Every map $(D^n, \partial D^n) \to (X, A)$ is homotopic to a constant map $D^n \to A$.

3. Due Monday, April 20

(1) Explain what the reduced version of the suspension of a pointed space X, denoted ΣX, should be; in other words, find a way to modify the suspension so that it has a well-defined basepoint. Show that this construction is functorial, and that the reduced suspension ΣX is homotopic to the ordinary suspension SX.

(2) Prove that if X is n-connected, then ΣX is $(n+1)$-connected. Use this fact to prove that in the suspension sequence

$$\pi_i(X) \to \pi_{i+1}(\Sigma X) \to \pi_{i+2}(\Sigma^2 X) \to \cdots$$

the maps eventually become isomorphisms.

4. Due Monday, May 4

(1) Sketch the proof that a $K(G, n)$ exists for any abelian group G and any $n \geq 2$.
(2) Give an example to show that the assumption that the spaces be simply connected is necessary in Corollary 4.33. (Hint: See example 2.38, and explain what happens for higher homotopy and homology groups.)

5. Due Monday, May 18

(1) Show that any fibration in the sequence

$$\cdots \Omega^2 B \to \Omega F \to \Omega E \to \Omega B \to F \to E \to B$$

gives the same long exact sequence of homotopy groups as the original fibration $F \to E \to B$.
(2) Show that you get, for any space X, a long exact sequence

$$\cdots \langle X, \Omega^2 B\rangle \to \langle X, \Omega F\rangle \to \langle X, \Omega E\rangle \to \langle X, \Omega B\rangle \to \langle X, F\rangle \to \langle X, E\rangle \to \langle X, B\rangle.$$

6. Due Monday, June 1

(1) Show that the map k' in the derived couple is well-defined.
(2) Show that the derived couple is an exact couple.