- (8) This problem is best proved directly, but it is pretty complicated, and I think it is better for you at this point to convince yourself that it makes sense conceptually. The idea is that if E is bounded, but f is not, then we must be able to find $p, q \in E$ such that $d(p, q) < \delta$ but $d(f(p), f(q)) > \varepsilon$. This fact then contradicts uniform continuity. Showing that being unbounded gives that we can find such a p and q is hard to show! With the final coming up soon, don't stress about proving this statement explicitly; just see if you believe it.

- (9) Suppose that $f\colon X \to Y$ is uniformly continuous. Then for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any p, q satisfying $d_X(p, q) < \delta$, we have $d_Y(f(p), f(q)) < \varepsilon$. Choose $E \subset X$ such that $\operatorname{diam}(E) < \delta$. Then we have that for any $p, q \in E$, $d_X(p, q) < \delta$. But then we know that $d_Y(f(p), f(q)) < \varepsilon$, by uniform continuity. But, since p and q were arbitrary, we know that $\operatorname{diam}(f(E)) < \varepsilon$.

  Conversely, suppose that for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\operatorname{diam}(E) < \varepsilon$ for every $E \subset X$ with $\operatorname{diam}(E) < \delta$. Consider the subsets $\{p, q\} \subset \mathbb{R}$ and $\{f(p), f(q)\} \subset \mathbb{R}$. Then $\operatorname{diam}\{p, q\} = d_X(p, q)$, and $\operatorname{diam}\{f(p), f(q)\} = d_Y(f(p), f(q))$, and from this case, applied to any p and q, we get the definition of uniform continuity.

- (10) Suppose that f is not uniformly continuous. Then for some $\varepsilon > 0$ there exist sequences $\{p_n\}$ and $\{q_n\}$ in X such that $d_X(p_n, q_n) \to 0$ but $d_Y(f(p_n), f(q_n)) > \varepsilon$. Since X is compact by assumption, we know that some subsequences of $\{p_n\}$ and $\{q_n\}$ converge to points of X; without loss of generality, we can just use these subsequences so that we have convergence. We also know by Theorem 4.14 that $f(X)$ is also compact. By Theorem 2.37, the sequences $\{f(p_n)\}$ and $\{f(q_n)\}$ have limits in $f(X)$. But, since $d_X(p_n, q_n) \to 0$, we must have $\{p_n\}$ and $\{q_n\}$ converging to the same point, say p, and therefore these limits in Y must both be $f(p)$, by Theorem 4.6. Thus, choose N such that $n \geq N$ implies that $d_Y(f(p_n), p) < \varepsilon/2$ and $d_Y(f(q_n), f(p)) < \varepsilon$. Then

  $$d_Y(f(p_n), f(q_n)) \leq d_Y(f(p_n), p) + d_Y(f(q_n), p) < \varepsilon$$

  which contradicts our assumption.

- (11) Given $\varepsilon > 0$, there exists $\delta > 0$ such that if $d_X(x_n, x_m) < \delta$, then $d_Y(f(x_n), f(x_m)) < \varepsilon$, since f is uniformly continuous. Since

$\{x_n\}$ is a Cauchy sequence, for this $\delta > 0$ there exists N such that if $m, n \geq N$, then $d_X(x_n, x_m) < \delta$. But then for this m, n we will then have $d_Y(f(x_n), f(x_n)) < \varepsilon$, showing that $\{f(x_n)\}$ is Cauchy.

- (14) Define $g(x) = f(x) - x$, which is continuous whenever f is. Then $g(0) = f(0) - 0 \geq 0$, and $g(1) = f(1) - 1 \leq 0$. If equality holds in either case, then we have our point such that $f(x) = x$. If not, then we can apply Theorem 4.23 to find a number c such that $g(c) = 0$. But, if $g(c) = 0$, then $f(c) - c = 0$, so $f(c) = c$.

- (15) Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is not monotonic but it is continuous. Then we can find a point $p \in \mathbb{R}$ and a neighborhood $E = (a, b)$ containing p such that either $f(p) \geq f(x)$ for all $x \in (a, b)$, or $f(p) \leq f(x)$ for all $x \in (a, b)$. In either case, we consider $f(E) \subset \mathbb{R}$ and $f(p) \in f(E)$. But, by our assumption on p (that $f(p)$ is either a maximum or a minimum in $f(E)$), there is no neighborhood of $f(p)$ contained in $f(E)$, so $f(p)$ is not an interior point and therefore $f(E)$ cannot be open, contradicting our assumption that f is an open map.

- (16) The function $[x]$ has a simple discontinuity at every integer $n \in \mathbb{Z}$, since $f(n-) = n-1$ but $f(n+) = n$. The function (x) also has a simple discontinuity at every $n \in \mathbb{Z}$ since $f(n-) = 1$ but $f(n+) = 0$ for any n.