Hyperfunctions and Spectral Zeta Functions of Laplacians on Fractals

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The key points

- Spectral zeta functions of Laplacians on fractals
- Factorization of the spectral zeta functions, usually in terms of well known zeta functions
The spectral zeta function

Definition

The spectral zeta function of a positive self-adjoint operator $L$ with compact resolvent (and hence, with discrete spectrum) is given by

$$\zeta_L(s) = \sum_{j=1}^{\infty} (\kappa_j)^{-s/2},$$

where the positive real numbers $\kappa_j$ are the eigenvalues of the operator written in nonincreasing order and counted according to their multiplicities.
The Laplacian on the unit interval

Consider the Dirichlet Laplacian $A = -\frac{d^2}{dx^2}$ on $[0,1]$. 

Eigenvalues: $\lambda = \pi^2 j^2$, $j = 1, 2, \ldots$.

The spectral zeta function is

$$
\zeta_A(s) = \sum_{j=1}^{\infty} (\pi^2 j^2)^{-s/2} \\
= \pi^{-s} \sum_{j=1}^{\infty} j^{-s} \\
= \pi^{-s} \zeta(s).
$$
Fractal string

**Definition**
A fractal string $\mathcal{L}$ is a countable collection of disjoint open intervals of length $\ell_k$.

**Definition**
The geometric zeta function of the fractal string is given by

$$\zeta_{\mathcal{L}}(s) = \sum_{k=1}^{\infty} (\ell_k)^s.$$
Consider $\mathcal{L} = \bigcup I_k$ with $|I_k| = \ell_k$ and $A = -\frac{d^2}{dx^2}$ be the Dirichlet Laplacian on $\mathcal{L}$. 

$\sigma(A) = \bigcup_{k=1}^{\infty} \sigma(A; I_k) = \{ \frac{\pi^2 n^2}{\ell_k^2} : n \geq 1, k \geq 1 \}$. 

The spectral zeta function of $A$ is 

$$\zeta_A(s) = \sum_{k,n \geq 1} \left( \frac{\pi^2 n^2}{\ell_k^2} \right)^{-s/2} = \pi^{-s} \sum_{k=1}^{\infty} \ell_k^s \sum_{n=1}^{\infty} n^{-s}.$$ 

**Theorem (Lapidus)**

$$\zeta_A(s) = \pi^{-s} \zeta_{\mathcal{L}}(s) \zeta(s), \text{ where } \zeta(s) \text{ is the Riemann zeta function.}$$
The Sierpinski Gasket

The Sierpinski gasket SG, is a classical example of a self-similar fractal on which the Laplacian is widely explored.
The normalized discrete Laplacian on $\Gamma_m$ is defined by

$$\Delta_m u(x) := \frac{1}{4} \sum_{x \sim y} (u(x) - u(y))$$

for $u : V_m \to \mathbb{R}$ with $u$ vanishing on $V_0$.

Then the Laplacian on the Sierpinski gasket is defined as a limit

$$\Delta u(x) = \lim_{m \to \infty} 5^m \Delta_m u(x).$$
Decimation method

In analyzing the eigenvalue spectrum of the Laplacian on SG, we first consider the eigenvalue equation associated with each $\Delta_m$:

$$\Delta_m u = \lambda u.$$ 

The decimation method is a process that relates the eigenvalues of successive graph Laplacians by the polynomial $R(z) = z(5 - 4z)$, in particular the inverse image:

- $R_-(z) = \frac{5 - \sqrt{25 - 16z}}{8}$
- $R_+(z) = \frac{5 + \sqrt{25 - 16z}}{8}$. 
Eigenvalue diagram

Operators | Eigenvalues
---|---
$-\Delta_0$ | 0 [1] $\frac{3}{2}$ [2]
$-\Delta_1$ | $\frac{5}{4}$ [1] $\frac{3}{4}$ [2] $\frac{3}{2}$ [3] $\frac{3}{2}$ [6] $\frac{5}{4}$ [1]
$-\Delta_2$ | $\frac{5}{4}$ [1] $\frac{3}{4}$ [3] $\frac{3}{2}$ [6] $\frac{3}{2}$ [15] $\frac{5}{4}$ [4]
$-\Delta_3$ | $\frac{5}{4}$ [1] $\frac{3}{2}$ [2] $\frac{3}{4}$ [6] $\frac{3}{2}$ [4] $\frac{5}{4}$ [4]
Factorization of the spectral zeta function

Definition

Let $R$ be a polynomial of degree $N$ satisfying $R(0) = 0$, $c := R'(0) > 1$, and with Julia set $\mathcal{J} \subset [0, \infty)$. Then the zeta function of $R$ of degree $N$ is defined by

$$\zeta_{R, z_0}(s) = \lim_{n \to \infty} \sum_{z \in R^{-n}\{z_0\}} (c^n z)^{-\frac{s}{2}}.$$

for $\text{Re}(s) > d_R := \frac{2 \log N}{\log c}$.

Theorem (Teplyaev)

The spectral zeta function of the Laplacian on $SG$ is

$$\zeta_{\Delta, \mu}(s) = \zeta_{R, \frac{3}{4}}(s) \frac{5^{-\frac{s}{2}}}{2} \left( \frac{1}{1 - 3 \cdot 5^{-\frac{s}{2}}} + \frac{3}{1 - 5^{-\frac{s}{2}}} \right) + \zeta_{R, \frac{5}{4}} \frac{5^{-s} \left( \frac{3}{1 - 3 \cdot 5^{-\frac{s}{2}}} - \frac{1}{1 - 5^{-\frac{s}{2}}} \right)}{2}$$

where $R(z) = z(5 - 4z)$. Furthermore, there exists $\epsilon > 0$ such that $\zeta_{\Delta, \mu}(s)$ has a meromorphic continuation for $\text{Re}(s) > -\epsilon$ with poles contained in

$$\left\{ \frac{2in\pi}{\log 5}, \frac{\log 9 + 2in\pi}{\log 5} : n \in \mathbb{Z} \right\}.$$
Now we discuss the factorization of the spectral zeta function of the fractal Sturm-Liouville operator on the half-line (a model introduced by C. Sabot).
We will extend the results on the previous slides to several complex variables.
Consider the two contraction mappings $\Psi_1$ and $\Psi_2$ on $I = [0, 1]$ defined by

$$\Psi_1(x) = \alpha x$$
$$\Psi_2(x) = 1 - (1 - \alpha)(1 - x).$$

- **Blow-up of the interval** $I$: $I_{<n>} = \Psi_1^{-n}(I) = [0, \alpha^{-n}]$.
- $I_{<n>} = \bigcup_{i_1 \ldots i_n} \Psi_{i_1 \ldots i_n}(I_{<n>})$ where $(i_1, \ldots, i_n) \in \{1, 2\}^n$. 

The self-similar measure on $I$: $m$ is a measure such that for all $f \in C([0, 1])$

$$\int_0^1 f dm = b \int_0^1 f \circ \Psi_1 dm + (1 - b) \int_0^1 f \circ \Psi_2 dm.$$ 

Define $H_{<0>} = \frac{d}{dm} \frac{d}{dx}$ with Dirichlet boundary condition by $H_{<0>} f = g$ on the domain 

$$\{ f \in L^2(I, m), \exists g \in L^2(I, m), f(x) = cx + d + \int_0^x \int_0^y g(z) dm(z) dy, f(0) = f(1) = 0 \}.$$ 

The operator $H_{<0>}$ is the infinitesimal generator associated with the Dirichlet form $(a, D)$ given by

$$a(f, g) = \int_0^1 f' g' dx$$

$$D = \{ f \in L^2(I, m) : f' \in L^2(I, dx) \}$$

$\forall f, g \in D.$
On the extension \( I_{<n>} \rightarrow I_{<\infty>} = \mathbb{R}_+ \)

Recall: \( I_{<n>} = \bigcup_{i_1 \ldots i_n} \Psi_{i_1 \ldots i_n}(I_{<n>}) \) where \((i_1, \ldots, i_n) \in \{1, 2\}^n\). We extend the definitions of the measure and the Dirichlet forms to \( I_{<n>} \).

- \( m_{<n>} \) is defined by \( \int_{I_{<n>}} f \, dm_{<n>} = (1 - \alpha)^{-n} \int_I f \circ \Psi_{1}^{-n} \, dm, \ f \in C(I_{<n>}) \).
- \((a_{<n>}, D_{<n>})\) is defined by
  \[
  a_{<n>}(f, f) = \int_0^{\alpha^{-n}} (f')^2 \, dx = \alpha^n a(f \circ \Psi_{1}^{-n}), \ \forall f \in D_{<n>}
  \]

\[
D_{<n>} = \{ f \in L^2(I_{<n>}, m_{<n>}) : f' \text{ exists and } f' \in L^2(I_{<n>}, dx) \}\]
The second order differential operator $H_{<n>} \to H_{<\infty>}$

We define: $\mathcal{D} = \{ f \in L^2(I_{<n>}, m_{<n>}), \exists g \in L^2(I_{<n>}, m_{<n>}), f(x) = ux + v + \int_0^x \int_0^y g(z)dm_{<n>}(z)dy, f(0) = f(\alpha^{-n}) = 0 \}$. 

**Definition**

For any $f \in \mathcal{D}$ we define the operator $H_{<n>} = \frac{-d}{dm_{<n>}} \frac{d}{dx}$ with Dirichlet boundary conditions on $I_{<n>}$ by $H_{<m>}f = g$.

The Sturm-Liouville operator $H_{<\infty>}$ on $[0, \infty)$ is viewed as a limit of the sequence of operators $H_{<n>} = -\frac{d}{dm_{<n>}} \frac{d}{dx}$ with Dirichlet boundary condition on $I_{<n>} = [0, \alpha^{-n}]$. 

Nishu Lal Joint work with M. Lapidus (UCR) Hyperfunctions and Spectral Zeta Functions of Laplacian
Consider the eigenvalue problem

\[ H_{<n>}f = \frac{-d}{dm_{<n>}} \frac{d}{dx} f = \lambda f \]

of the Sturm-Liouville operator with Dirichlet boundary condition on \( I_{<n>} \).
Renormalization map

The study of the eigenvalue problem revolves around a map, called the renormalization map, which is initially defined on a space of quadratic forms associated with the fractal. The propagator of the differential equation is very useful in producing this rational map,

\[ \rho([x, y, z]) = [x(x + \delta^{-1}y) - \delta^{-1}z^2, \delta y(x + \delta^{-1}y) - \delta z^2, z^2] \]

defined on \( \mathbb{P}^2(\mathbb{C}) \).

Here, \([x, y, z]\) denote the homogeneous coordinate of a point in \( \mathbb{P}^2(\mathbb{C}) \), where \((x, y, z) \in \mathbb{C}^3\) is identified with \((\beta x, \beta y, \beta z)\) for any \(\beta \in \mathbb{C}, \beta \neq 0\).
The invariant curve

We introduce the invariant curve $\phi(x)$ which is holomorphic on $\mathbb{C}$ such that for all $\lambda \in \mathbb{C}$

$$\rho(\phi(\lambda)) = \phi(\gamma \lambda)$$
The spectrum of $H_{<n>}$ on $l_{<n>}$

The dynamics of the renormalization map $\rho$ plays a key role in calculating the spectrum of the operators $H_{<n>}$. 

- Let $D = \{ [x, y, z] : x + \delta^{-1} y = 0 \}$.
- The set $D$ is part of the Fatou set of $\rho$.
- Let $S$ be the time intersections of the curve $\phi(\gamma^{-1}\lambda)$ with $D$,
  \[ S = \{ \lambda \in \mathbb{C} : \phi(\gamma^{-1}\lambda) \in D \}. \]
The spectrum of $H_{<0>}$

**Theorem (Sabot)**

Let $S_p = \gamma^p S$ with $p \in \mathbb{Z}$. Then the spectrum of $H_{<0>}$ on $I = I_{<0>}$ is $\bigcup_{p=0}^{\infty} S_p$ and the spectrum of $H_{<\infty>}$ on $\mathbb{R}^+$ is $\bigcup_{p=-\infty}^{\infty} S_p$. Moreover, for any $n \geq 0$, the spectrum of $H_{<n>}$ is equal to $\bigcup_{p=-n}^{\infty} S_p$.

The set $S$ is non-empty and contained in $\mathbb{R}_+$. We arrange the elements of $S$ as $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots$. 
The spectrum of $H_{<\infty}$

The eigenvalue diagram with each $\lambda_j \in S$ has the following form:

\[
\begin{array}{cccccc}
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \cdots \\
\gamma \lambda_1 & \gamma \lambda_2 & \gamma \lambda_3 & \gamma \lambda_4 & \cdots \\
\gamma^2 \lambda_1 & \gamma^2 \lambda_2 & \gamma^2 \lambda_3 & \gamma^2 \lambda_4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\end{array}
\]
The zeta function associated to the renormalization map

We introduce a multivariable analog of the polynomial zeta function.

**Definition (L-Lapidus)**

We define the *zeta function of the renormalization map* \( \rho \) to be

\[
\zeta_\rho(s) = \sum_{p=0}^{\infty} \sum_{\lambda \in \mathbb{C}: \rho^p(\phi(\gamma^{-(p+1)}\lambda)) \in D} (\gamma^p\lambda)^{-\frac{s}{2}},
\]

for \( \text{Re}(s) \) sufficiently large.

\[ (1) \]
The spectral zeta function of $H_{<0>}$

Recall that given an integer $n \geq 0$, the spectral zeta $\zeta_{H_{<n>}}(s)$ of $H_{<n>}$ on $[0, \alpha^{-n}]$ is

$$
\zeta_{H_{<n>}}(s) = \sum_{\lambda \in S} \sum_{p=-n}^{\infty} (\gamma^p \lambda)^{-\frac{s}{2}}.
$$

**Theorem (L-Lapidus)**

The zeta function $\zeta_\rho(s)$ of the renormalization map $\rho$ is equal to the spectral zeta function $\zeta_{H_{<0>}}(s) = \sum_{\lambda \in S} \sum_{p=0}^{\infty} (\gamma^p \lambda)^{-\frac{s}{2}}$ of $H_{<0>}(s)$:

$$
\zeta_\rho(s) = \zeta_{H_{<0>}}(s).
$$
The factorization formulas for $\zeta_{<n>}$

**Proposition (L-Lapidus)**

For $n \geq 0$ and Re$(s)$ sufficiently large and positive, we have

$$\zeta_{H_{<n>}}(s) = \frac{(\gamma^n)^{s/2}}{1 - \gamma^{-s/2}} \sum_{j=1}^{\infty} (\lambda_j)^{-s/2}. \quad (2)$$

Hence, $\zeta_{H_{<n>}}(s) = \frac{(\gamma^n)^{s/2}}{1 - \gamma^{-s/2}} \zeta_S(s)$, where $\zeta_S(s) := \sum_{j=1}^{\infty} (\lambda_j)^{-s/2}$ (for Re$(s)$ large enough) or is given by its *meromorphic continuation* thereof. In the sequel, $\zeta_S(s)$ is called the *geometric zeta function of the generating set S*. 
The spectral zeta function $\zeta_{<\infty}$

For the spectral zeta function $\zeta_{<\infty}$ of $H_{<\infty}$, we have the following:

$$
\zeta_{\infty}(s) = \sum_{j=1}^{\infty} \sum_{p=-\infty}^{\infty} (\gamma^p \lambda_j)^{-\frac{s}{2}}
$$

$$
= \left( \frac{\gamma^{\frac{s}{2}}}{1 - \gamma^{\frac{s}{2}}} + \frac{1}{1 - \gamma^{-\frac{s}{2}}} \right) \sum_{j=1}^{\infty} (\lambda_j)^{-\frac{s}{2}}
$$

$$
= \left( \frac{1}{1 - \gamma^{-\frac{s}{2}}} - \frac{1}{1 - \gamma^{-\frac{s}{2}}} \right) \zeta(s).
$$

The geometric part $\frac{1}{1 - \gamma^{-\frac{s}{2}}} - \frac{1}{1 - \gamma^{-\frac{s}{2}}}$ of the product structure is equal to zero. However, it represents an example of a hyperfunction known as the Dirac $\delta$-function.
The hyperfunctions are the distributional generalization of analytic functions.

A hyperfunction \( f = [F_+, F_-] \) consists of two analytic functions, \( F_+(z) \) and \( F_-(z) \), which are analytic in the upper and the lower half-planes respectively, such that the following limit makes sense

\[
\lim_{\epsilon \to 0} \left[ F_+(x + i\epsilon) - F_-(x - i\epsilon) \right].
\]
The Dirac delta hyperfunction

The Dirac delta function on the real line $\mathbb{R}$ is defined by

$$\delta_{\mathbb{R}}(z) := \left[ -\frac{1}{2\pi iz}, -\frac{1}{2\pi iz} \right].$$

By definition, for $x \neq 0$,

$$\delta(x) = \lim_{\epsilon \to 0^+} \left( F_+(x + i\epsilon) - F_-(x - i\epsilon) \right)$$

$$= \lim_{\epsilon \to 0^+} \left( -\frac{1}{2\pi i(x + i\epsilon)} - \frac{1}{2\pi i(x - i\epsilon)} \right)$$

$$= \lim_{\epsilon \to 0^+} \frac{\epsilon}{\pi(x^2 + \epsilon^2)} = 0.$$ 

For $x = 0$, however, the above limit does not exist, and this is the point at which the delta ‘function’ has an isolated singularity.
Factorization of $\zeta_{H^{<\infty}}$

In the case of the spectral zeta function $\zeta_{H^{<\infty}}$, the geometric part of the spectral zeta function can be represented by the Dirac delta function with a suitable change of variable.

**Theorem (L-Lapidus)**

The factorization of the spectral zeta function $\zeta_{H^{<\infty}}(s)$ of $H^{<\infty}$ is given by

$$\zeta_{H^{<\infty}}(s) = \zeta_S(s) \cdot \delta_T(w),$$

where $w := \gamma^{-\frac{s}{2}}$ and $\delta_T(w) = [\delta_T^+(w), \delta_T^-(w)]$ is the Dirac delta hyperfunction on the unit circle $\mathbb{T}$.

(Note that $|w| < 1$ for $Re(s) > 0$ and $|w| > 1$ for $Re(s) < 0$, since $\gamma \geq 4$ implies that $\log \gamma > 0$.)
A representation of the Riemann zeta function

When \( \alpha = \frac{1}{2} \), we obtain the usual Laplacian on the unit interval.

**Theorem (L-Lapidus)**

When \( \alpha = \frac{1}{2} \), the Riemann zeta function \( \zeta \) is equal (up to a trivial factor) to the zeta function \( \zeta_\rho \) associated with the renormalization map \( \rho \) on \( \mathbb{P}^2(\mathbb{C}) \). More specifically, we have

\[
\zeta(s) = \pi^s \zeta_\rho(s) = \frac{\pi^s}{1 - 2^{-s}} \zeta_S(s). \tag{4}
\]

This is an extension to several complex variables of the result by A. Teplyaev which states that the Riemann zeta function can be written in terms of the zeta function of a quadratic polynomial of one complex variable.
We now extend the finite Sierpinski gasket to the infinite (or unbounded) Sierpinski gasket. Let $k = \{k_n\}_{n \geq 1}$ be a fixed sequence, with $k_n \in \{1, 2, 3\}$ for all $n \geq 1$. We construct a sequence $SG^{(n)} = \Psi_{k,n}^{-1}(SG)$, where $\psi_{k,n} = \psi_{k_1 ... k_n} := \psi_{k_n} \circ \ldots \circ \psi_{k_1}$. The infinite Sierpinski gasket is then defined by

$$SG^{(\infty)} = \bigcup_{n=0}^{\infty} SG^{(n)},$$

viewed as a blow-up of $SG$. The mth pre-gasket approximating $SG^{(n)}$ and $SG^{(\infty)}$ are $V_m^{(n)} = \psi_{k,n}^{-1}(V_{n+m})$ and $V_m^{(\infty)} = \bigcup_{n=0}^{\infty} \psi_{k,n}^{-1}(V_{n+m})$, respectively.
We next define the Laplacian $\Delta^{(n)}$ on $SG^{(n)}$ as follows:

$$\Delta^{(n)} u = f \in L^2(SG^{(n)}, \mu) \text{ iff } E_{SG^{(n)}}(u, v) = \int_{SG^{(n)}} \Delta^{(n)} u v d\mu,$$

where $E_{SG^{(n)}}$ is a scaled copy of $E_{n+m}$ on $V_{n+m}$ for the finite Sierpinski pre-gasket.

The pointwise Laplacian $\Delta^{(\infty)}$ on $SG^{(\infty)}$ can then be defined by the following pointwise limit:

$$5^n \Delta^{(n)} u \to \Delta^{(\infty)} u \text{ as } n \to \infty.$$
Let \( R(z) = z(5 - 4z) \). Then the spectrum of the self-adjoint operator \( \Delta^{(\infty)} \) acting on \( L^2(SG^{(\infty)}, \mu) \) is pure point and the set of compactly supported eigenfunctions is complete. Furthermore, the set of eigenvalues is given by \( \bigcup_{n=-\infty}^{\infty} 5^n R\{\Sigma\} \), where \( \Sigma = \{\frac{3}{2}\} \cup \left(\bigcup_{j=0}^{\infty} R^{-j}\{\frac{3}{4}\}\right) \cup \left(\bigcup_{j=0}^{\infty} R^{-j}\{\frac{5}{4}\}\right) \) is the set of eigenvalues of the Laplacian \( \Delta_\mu \) on the finite SG, \( R(z) := \lim_{m \to \infty} 5^m R^{-m}(z) \) and \( R^{-m} \) is the branch of the mth inverse iterate of \( R \) that passes through the origin.

In particular, the spectrum of \( \Delta^{(\infty)} \) has the following form:

\[
\bigcup_{n=-\infty}^{\infty} \bigcup_{j=0}^{\infty} 5^n R(R^{-j}(z_0)),
\]

(5)

where \( z_0 = \frac{3}{4}, \frac{5}{4} \). Every eigenvalue \( \lambda \) of \( \Delta^{(\infty)} \) can be expressed as \( \lambda = 5^n \lim_{m \to \infty} 5^m R^{-m}(z_m) \) for some \( n \in \mathbb{Z} \), with \( z_m \) in the spectrum \( \sigma(\Delta_m) \) of the finite mth Sierpinski pre-gasket.
Theorem (L-Lapidus)

The spectral zeta function \( \zeta_{\Delta(\infty)} \) of the Laplacian \( \Delta^{(\infty)} \) on the infinite Sierpinski gasket \( SG^{(\infty)} \) is given by

\[
\zeta_{\Delta(\infty)}(s) = \delta_T(5^{-\frac{s}{2}})\zeta_{\Delta\mu}(s),
\]

where \( \delta_T \) is the Dirac hyperfunction and \( \zeta_{\Delta\mu} \) is the spectral zeta function of the Laplacian on the finite \( SG \).
Thank you very much!


