Non-Gaussian Dynamics, Nonlocal Operators & Applications

Jinqiao Duan
Institute for Pure and Applied Mathematics
Los Angeles
&
Dept. of Applied Math, Illinois Institute of Technology
Chicago
mypages.iit.edu/~duan
Outline

1. Motivation

2. Impressions of Stochastic Dynamics

3. Non-Gaussian Dynamics
   - Escape probability
   - Mean exit time
   - Bifurcation

4. Conclusions
Dynamical systems

- **Deterministic dyn systems:** Geometric views on solutions
  Ordinary diff eqns (ODEs)
  Partial diff eqns (PDEs)

- **Stochastic dyn systems:** Geometric views on solutions
  Stochastic diff eqns (SDEs)
  Stochastic partial diff eqns (SPDEs)
1940s: Ito stochastic calculus

1970s: Stochastic differential equations (SDEs)
Ikeda-Watanabe, Arnold, Friedman

1980s: Stochastic flows, cocycles
Elworthy, Baxendale, Bismut, Ikeda, Kunita, ...

1990s: Dynamical systems approaches for SDEs
L. Arnold, ......

Related development:
- Ergodic theory
- Statistical mechanics
Dynamical systems with noise!
- Environmental or intrinsic fluctuations! (seen in observations)
- **Gaussian noise** or **non-Gaussian noise**
- Brownian motion or $\alpha$–stable Lévy motions
Scientific observations: Many independent measurements and then “averaging"

Where do Gaussian random variables come from? 
$X_1, X_2, \cdots, X_n$ are independent, identically distributed (iid) random variables with finite mean $\mu$ and variance $\sigma^2$

Central limit theorem: Gaussian random variables come from “averaging"

$$\lim_{n \to \infty} \frac{X_1 + \cdots + X_n - n\mu}{\sigma \sqrt{n}} = X \sim \mathcal{N}(0, 1) \quad \text{in distribution}$$

Namely,

$$\lim_{n \to \infty} \mathbb{P}(\frac{X_1 + \cdots + X_n - n\mu}{\sigma \sqrt{n}} \leq x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-x^2/2} \, dx$$
Scientific observations:
Many independent measurements and then "averaging"

Where do stable random variables come from?

\(X_1, X_2, \ldots, X_n\) are independent, identically distributed (iid) random variables (whose mean and variance may be infinite)

Definition:
\(X\) is a stable random variable if it is a limit (in distribution) of an averaging sequence of \(X_i\)’s:

\[
\lim_{n \to \infty} \frac{X_1 + \cdots + X_n - b_n}{a_n} = X \quad \text{in distribution}
\]

for some constants \(a_n, b_n \, (a_n \neq 0)\)

Notation: \(X \sim S_\alpha(\sigma, \beta, \mu)\)
Gaussian vs. Non-Gaussian random variables

Gaussian random variable $X(\omega)$:

Probability density function (PDF) \[ f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \]

\[ \mathbb{P}(X \leq x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx \]

Non-Gaussian random variable $X(\omega)$:

Probability density function (PDF) \[ f(x) \geq 0, \quad \int_{\mathbb{R}} f(x) \, dx = 1. \]

\[ \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f(x) \, dx \]

Gaussian process & non-Gaussian process
Also called a normal random variable: $X \sim \mathcal{N}(\mu, \sigma^2)$
Prob density function for a non-Gaussian random variable
• Brownian motion is defined in terms of Gaussian r. v.

• $\alpha-$stable Lévy motion is defined in terms of stable r. v.
Brownian motion $B_t$: A Gaussian process

- Independent increments: $B_{t_2} - B_{t_1}$ and $B_{t_3} - B_{t_2}$ independent
- Stationary increments with $B_t - B_s \sim N(0, t - s)$
  In particular, $B_t \sim N(0, t)$
- Continuous sample paths, but nowhere differentiable

Reference:
I. Karatzas and S. E. Shreve,
Brownian Motion and Stochastic Calculus
A sample path for Brownian motion $B_t(\omega)$

Figure: Continuous path, but nowhere differentiable
**α-stable Lévy Motion \( L_t \): A non-Gaussian process**

**Definition:** \( α \)-stable Lévy motion \( L_t(ω) \) with \( 0 < α < 2 \):

1. \( L_0 = 0, \) a.s.;
2. \( L_t \) has independent increments;
3. Stationary increments \( L_t - L_s \sim S_α(\left| t - s \right|^\frac{1}{α}, \beta, 0) \);

**Note 1.** Paths are stoch. continuous (right continuous with left limit; countable jumps): \( L_t \rightarrow L_s \) in prob. as \( t \rightarrow s \)

**Note 2.** \( α = 2 \): Brownian motion \( B_t \) and

\( B_t - B_s \sim S_2(\left| t - s \right|^{\frac{1}{2}}, 0, 0) \)

**Note 3.** When \( β = 0 \), Lévy jump measure:

\[
ν_α(du) = \frac{C_α}{\left| u \right|^{1+α}}(du),
\]

\( α \)-stable symmetric Lévy Motion. \( (C_α = \frac{αΓ((d+α)/2)}{2^{1-α}\sqrt{π}} Γ(1-α/2)} \)
$\alpha$-stable symmetric Lévy Motion $L_t$

Lévy jump measure: $\nu_\alpha(du) = c_\alpha \frac{1}{|u|^{1+\alpha}}(du)$

$0 < \alpha < 2$: Lévy motion $L_t$

$\alpha = 2$: Brownian motion $B_t$

Heavy tail for $0 < \alpha < 2$: **Power law** (non-Gaussian)

$$\mathbb{P}(|L_t| > u) \sim \frac{1}{u^\alpha}$$

Light tail for $\alpha = 2$: **Exponential law** (Guassian)

$$\mathbb{P}(|B_t| > u) \sim \frac{e^{-u^2/2}}{\sqrt{2\pi u}}$$

Reference:

D. Applebaum — **Lévy Processes and Stochastic Calculus**
A sample path for $\alpha-$stable Lévy motion $L_t(\omega)$: Jumps

**Figure:** $\alpha-$stable Lévy motion: $\alpha = 0.25$
A sample path for $\alpha$–stable Lévy motion $L_t(\omega)$: Jumps

**Figure:** $\alpha$–stable Lévy motion: $\alpha = 0.75$
A sample path for $\alpha$–stable Lévy motion $L_t(\omega)$: Jumps

Figure: $\alpha$–stable Lévy motion: $\alpha = 1.9$
### Summary on comparison of BM and $\alpha$–stable LM

**Ting Gao 2011**

<table>
<thead>
<tr>
<th>Brownian Motion</th>
<th>$\alpha$–stable Levy Motion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Independent increment</td>
<td>Independent increment</td>
</tr>
<tr>
<td>Stationary increment $B_t - B_s \sim S_2(</td>
<td>t - s</td>
</tr>
<tr>
<td>Continuous sample paths</td>
<td>Stoch continuous paths (&quot;jumps&quot;)</td>
</tr>
<tr>
<td>Triplet $(a, d, 0)$</td>
<td>Triplet $(a, 0, \nu_\alpha)$</td>
</tr>
</tbody>
</table>
Properties of paths of \( \alpha \)-stable Lévy motion \( L_t \)

(i) Countable and dense jumps in time

(ii) Right continuous with left limit at each jump time
Why Lévy motion $L_t(\omega)$: Jumps or flights

- Abrupt climate change such as Dansgaard-Oeschger events.

  *Ditlevsen 1999: Ice record for temperature*

- Diffusion of tracers in rotating annular flows: Pauses near coherent structures & jumps or “flights” in between PDF for flight times: Power laws

  *Swinney et al. 1995*

- Data in biology and other areas

  *Shlesinger et al.: Lévy Flights and Related Topics in Physics, 1995*

- Woyczynski, Lévy processes in the physical science, 2001

- Financial data
What is noise?

**Stationary process** $X(t)$: mean $\mathbb{E}X(t)$ is a constant and the autocorrelation $\mathbb{E}(X(t_1)X(t_2))$ depends only on the time lag $t_2 - t_1$.

**Noise**: $\eta_t$

- A stationary stochastic process
- Mean $\mathbb{E}\eta_t = 0$
- Covariance $\mathbb{E}(\eta_t\eta_s) = K c(t - s)$ for all $t$ and $s$, $K$ is a constant

**White noise**: $c(t - s) = \delta(t - s)$ Dirac Delta function

**Colored noise**: Non-white noise
Gaussian noise: $\frac{d}{dt}B_t$

Brownian motion $B_t$ is a Gaussian process with stationary (and also independent) increments, together with mean zero $\mathbb{E} B_t = 0$ and covariance $\mathbb{E}(B_tB_s) = t \wedge s = \min\{t, s\}$.

**Increments:** $B_{t+\Delta t} - B_t \approx \Delta t \dot{B}_t$ are stationary

**Mean:** $\mathbb{E} \dot{B}_t \approx \mathbb{E} \frac{B_{t+\Delta t} - B_t}{\Delta t} = 0$

**Covariance:**
By the formal formula $\mathbb{E}(\dot{X}_t\dot{X}_s) = \partial^2 \mathbb{E}(X_tX_s)/\partial t\partial s$, we see that

$$\mathbb{E}(\dot{B}_t\dot{B}_s) = \partial^2 \mathbb{E}(B_tB_s)/\partial t\partial s = \partial^2 (t \wedge s)/\partial t\partial s = \delta(t - s).$$

**Made rigorous:** Theory of Generalized Functions
Why is it called white noise?

The spectral density function for \( \dot{B}_t \), i.e., the Fourier transform \( \mathcal{F} \) for its covariance function \( \mathbb{E}(\dot{B}_t \dot{B}_s) \), is constant

\[
\mathcal{F}(\mathbb{E}(\dot{B}_t \dot{B}_s)) = \mathcal{F}(\delta(t - s)) = \frac{1}{2\pi}.
\]

Thus \( \eta_t = \dot{B}_t \) is taken as a mathematical model for white noise.

**Analogy**: White light in **Optics**
Non-Gaussian noise: $\frac{d}{dt} L_t$

*Lee & Shih 2008*
Gaussian & Non-Gaussian Noise

- Gaussian noise: $\frac{d}{dt} B_t$
- Non-Gaussian noise: $\frac{d}{dt} L_t$
Dynamics under Noise

Dynamical Systems

Gaussian/Brownian motion
Non-Gaussian/Levy motion

Input
Output
Impact of Gaussian noise on solutions

\[ dX_t = (-X_t + X_t^3)dt + \varepsilon dB_t, \quad X_0 = 0.5. \]

A solution path:

**Figure:** A solution path with “wigglings” when \( \varepsilon = 0.05 \), on top of the
Impact of non-Gaussian noise on solutions

\[ dX_t = (-X_t + X_t^3)dt + \varepsilon dL_t, \quad X_0 = 0.5. \]

A solution path: \( \alpha = 0.85 \)

**Figure:** A solution path with “wigglings” when \( \varepsilon = 0.05 \), on top of the
Impact of non-Gaussian noise on solutions

\[ dX_t = (-X_t + X_t^3)dt + \varepsilon dL_t, \quad X_0 = 0.5. \]

A solution path: \( \alpha = 1.5 \)

**Figure:** A solution path with “wigglings” when \( \varepsilon = 0.05 \), on top of the
Impact of noise on pathwise uniqueness of solutions

\[ \dot{x} = f(x) \]: a sufficient condition for uniqueness of solutions is the local Lipschitz continuity condition for vector field \( f \). Without this condition, the uniqueness is often violated.

However, for \( \dot{x} = f(x) + \dot{L}_t \) with \( f \) being Borel measurable and bounded (no Lipschitz condition), the uniqueness holds.

**Priola 2010**: Nonlocal PDE argument!
What is dynamics?

- **Look at** solutions collectively, as time goes on
- **Examine** solution mappings, as time goes on
- **Discover** structures as stepping stones for understanding
Deterministic dynamical systems

Linear system:

\[ x' = Ax \quad , \quad x(0) = x_0 \]

Solution mapping (Matrix exponential):

\[ \varphi(t, x_0) \triangleq e^{At} x_0 \]

“Flow” property:

\[ \varphi(t + s, x_0) = e^{A(t+s)} x_0 = \varphi(t, \varphi(s, x_0)) \]

Nonlinear system:

\[ x' = f(x) \quad , \quad x(0) = x_0 \]

Solution mapping:

\[ \varphi(t, x_0) \]

“Flow” property:

\[ \varphi(t + s, x_0) = \varphi(t, \varphi(s, x_0)) \]
Stochastic dynamical systems

Langevin equation:

$$dx = -x dt + dB_t, \quad x(0) = x_0$$

Random solution mapping:

$$\varphi(t, x_0, \omega) = x_0 e^{-t} + \int_0^t e^{-(t-s)} dB_s(\omega)$$

Linear stochastic systems: Random matrices
How to understand (non-Gaussian) stochastic dynamics?

- **Topological methods**: Invariants (e.g., Poincare index, Conley index)
- **Geometric methods**: Invariant structures (e.g., invariant manifolds, invariant foliations)
- **Functional analytical methods**: Nonlocal PDEs!
Topological methods: Invariants

Still in infancy!

**Conley index:** Liu 2008; Chen-Duan-Fu 2010

Difficulty due to the very nature of random invariant sets!
A caveat!

**Topological:** *Brouwer fixed point theorem*

A continuous map from a convex compact subset $K$ to $K$ itself has a fixed point.

Does NOT hold for random mappings — Some continuous random mappings on compact intervals do not have random invariant points

*Ochs and Oseledets* 1999
How to understand (non-Gaussian) stochastic dynamics?

- **Topological methods**: Invariants (e.g., Poincare index, Conley index)

- **Geometric methods**: Invariant structures (e.g., invariant manifolds, invariant foliations)

- **Functional analytical methods**: Nonlocal PDEs!
**Geometric methods: Invariant manifolds**

Banach fixed point theorem

A contraction mapping has a unique fixed point.

Invariant manifolds (Liapunov-Perron method)

Still hold for random mappings — contraction in mean

Schmalfuss 1997
Basic definitions

- **Random invariant set** $M(\omega)$:
  \[
  \varphi(t, \omega, M(\omega)) = M(\theta_t \omega) \text{ for } t \in \mathbb{R}
  \]

- **Stable manifold**:
  If we can represent an invariant set as a graph of a Lipschitz mapping
  \[
  h^s(\cdot, \omega) : \mathcal{H}^s \rightarrow \mathcal{H}^u
  \]
  such that
  \[
  \mathcal{W}^s(\omega) = \{ \xi + h^s(\xi, \omega) | \xi \in \mathcal{H}^s \}
  \]
  then $\mathcal{W}^s(\omega)$ is called a Lipschitz stable manifold.

- **Unstable manifold**:
  Similar
Theorem: Impact of noise on stable manifold

- **Assumptions:** \( \frac{du}{dt} = Au + F(u) + \epsilon u \circ \dot{L}_t \)

  (i) Linear part \( A \): exponential dichotomy ("saddle property")

  (ii) Nonlinear part \( F(u) \): twice continuously Frechet differentiable with respect to \( u \)

  (iii) Gap condition: \( K L_{\epsilon} \left( \frac{1}{\alpha} - \frac{1}{\beta} \right) < 1 \)
Theorem: Impact of noise on stable manifold

- **Random stable manifold:** $\tilde{W}^s(\omega) = \{\xi + h^s(\xi, \omega) | \xi \in H^s\}$

- **Approximating stable manifold:**
  When $\epsilon$ is sufficiently small, $W^s$ can be expressed as

  \[
  h^s = h^{(d)}(\xi) + \epsilon h^{(1)}(\xi, \omega) + R_s
  \]

  where $\|R_s\| \leq C(\omega)\epsilon^2$ with $C(\omega) < \infty$, $h^{(d)}$ is deterministic stable manifold, and

  \[
  h^{(1)}(\xi, \omega) = \int_0^0 e^{-As}\{[Z(\theta_r(\omega))dr

  + Z(\theta_s(\omega))] P^u F(u_0) + P^u F^u_0(s)[u_1(s) - Z(\theta_s(\omega))u_0(s)]\}\ ds
  \]

  **Note:** $Z(\omega)$ is the stationary solution of

  \[
  dZ(t) + Z(t)dt = dB(t)
  \]
Example:
What is impact of noise on invariant manifolds?

Consider a SDE system

\[
\begin{cases}
\dot{x} = -x + \epsilon x \circ \dot{B}_t, \\
\dot{y} = y + x^2 + \epsilon y \circ \dot{B}_t,
\end{cases}
\]

\[0 < \epsilon \ll 1\]

\(\tilde{W}^s\): Stable manifold in a neighborhood around (0, 0)

\(\epsilon = 0\): Deterministic stable manifold is

\[
W^s = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \left| y = -\frac{x^2}{3} \right. \right\}
\]
Deterministic stable manifold: $\varepsilon = 0$

\[
\begin{align*}
\dot{x} &= -x \\
\dot{y} &= y + x^2
\end{align*}
\]

Stable manifold $W^s$: $y = -\frac{1}{3}x^2$
Random stable manifold: $\varepsilon = 0.01$
Random stable manifold: $0 < \varepsilon \ll 1$

Random stable manifold:

\[
\tilde{W}^s = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| y = -\frac{x^2}{3} - \varepsilon \frac{x^2}{3} \left( \int_0^\infty e^{-3\tau} dB_\tau \right) + O(\varepsilon^2) \right\}
\]

Remarks:
With accuracy of $O(\varepsilon^2)$,

(i) **Mean** of random stable manifold is just the deterministic stable manifold

(ii) **Variance** of the random stable manifold is $\frac{1}{54} x^4$

Sun, Duan & Li: 2010

*Open problem*: Non-Gaussian Lévy noise
How to understand (non-Gaussian) stochastic dynamics?

- **Topological methods: Invariants** (e.g., Poincare index, Conley index)

- **Geometric methods: Invariant structures** (e.g., invariant manifolds, invariant foliations)

- **Functional analytical methods: Nonlocal PDEs!**
Functional analytical methods: **Nonlocal PDEs!**

**Main Ideas:**

(i) Solutions of stochastic systems are Markov processes

(ii) Markov processes $\rightarrow$ Semigroups

(iii) Semigroups’ generators $A$: **Nonlocal operators!**

(iv) Non-Gaussian Lévy noise $\sim$ **Nonlocal operators!**
Generators of Markov process $X_t$: $X_0 = x$

**Semigroup**: For observable $\varphi$

$P_t \varphi(x) \triangleq \mathbb{E} \varphi(X_t)$

$P_{t+s} = P_t P_s$

**Generator of semigroup**: Derivative of semigroup at time 0

$A \varphi(x) \triangleq \frac{d}{dt} \Big|_{t=0} P_t \varphi(X_t)$
Example: Generators of $B_t$ & $L_t$

Brownian motion $B_t$: Generator is $\frac{1}{2} \Delta$

$\alpha$-stable Lévy motion: Generator is $-K_\alpha (-\Delta)^{\frac{\alpha}{2}}$

Nonlocal operator:

$$\int_{\mathbb{R}^d \setminus \{0\}} [u(x + y) - u(x) - I_{\{||y|| < 1\}} yu'(x)] \nu_\alpha(dy) = -K_\alpha (-\Delta)^{\frac{\alpha}{2}}$$

Applebaum 2009
Functional analytical methods: **Nonlocal PDEs!**

- Escape probability
- Mean exit time
- Bifurcation
Escape probability from a domain $D$

$$dX_t = f(X_t)dt + g(X_t)dL_t, \quad X_0 = x \in D$$

$p(x)$: Likelihood that a “particle” first escapes $D$ and lands in a domain $U$

**Figure:** Escape probability for SDEs driven by Lévy motions: an open annular domain $D$, with its inner part $U$ (which is in $D^c$) as a target domain
Escape probability from a domain $D$

$$dX_t = f(X_t)dt + g(X_t)dL_t, \quad X_0 = x \in D$$

$p(x)$: Likelihood that a “particle” first escapes $D$ and lands in a domain $U$

**Figure:** Escape probability for SDEs driven by Lévy motions: a general open domain $D$, with a target domain $U$ in $D^c$
Recall: Classical harmonic functions

\[ \Delta h(x) = 0 \]

But \( \Delta \) is the generator of Brownian motion \( B_t \)

\( h(x) \): Harmonic function with respect to Brownian motion
What is a harmonic function with respect to $\alpha$–stable Lévy motion?
Brownian motion $B_t$: Generator is $\frac{1}{2} \Delta$

$\alpha$–stable Lévy motion: Generator is $-K_\alpha (-\Delta)^{\frac{\alpha}{2}}$
Harmonic function with respect to $\alpha$–stable Lévy motion:

$$-(-\Delta)^{\alpha/2} h(x) = 0$$
General harmonic functions

Harmonic function with respect to a Markov process with generator $A$:

$Ah(x) = 0$
Escape probability vs. harmonic functions

\[ dX_t = f(X_t)dt + g(X_t)dL_t, \quad X_0 = x \in D \]

For

\[ \varphi(x) = \begin{cases} 
1, & x \in U, \\
0, & x \in D^c \setminus U,
\end{cases} \]

\[
\mathbb{E}[\varphi(X_{\tau_{D^c}}(x))] = \int_{\{\omega : X_{\tau_{D^c}}(x) \in U\}} \varphi(X_{\tau_{D^c}}(x))dP(\omega) \\
+ \int_{\{\omega : X_{\tau_{D^c}}(x) \in D^c \setminus U\}} \varphi(X_{\tau_{D^c}}(x))dP(\omega)
\]

\[ = \mathbb{P}\{\omega : X_{\tau_{D^c}}(x) \in U\} \\
= p(x). \]

Left hand side is a harmonic function: Liao 1989
Escape probability from a domain $D$

Liao 1989; Kan, Qiao & Duan, 2011

\[
dX_t = f(X_t)dt + g(X_t)dL_t, \quad X_0 = x \in D
\]

$p(x)$: Likelihood that a “particle” first escapes $D$ and lands in a domain $U$

**Theorem**

*Escape probability is solution of Balayage-Dirichlet problem*

\[
\begin{align*}
A p &= 0, \\
|p|_U &= 1, \\
|p|_{D^c \setminus U} &= 0,
\end{align*}
\]

where $A$ is the generator for this system.
Example: A tumor growth system

Hao, Gao, Duan & Xu 2012
A reduced version of a Michaelis-Menten model for tumor cell population density $X_t$: Fiasconaro et. al. 2008

$$dX_t = X_t(1 - \theta X_t) - \frac{\beta X_t}{1 + X_t} + \varepsilon dL_t^\alpha, \quad X_0 = x$$

Growth rate of tumor tissue — environmental factors, such as the supply of nutrients, the immunological state of the host, chemical agents, temperature, and radiations

$\theta = 0.1, \beta = 3.0$

Deterministic system has two steady stable states: 0 and 5
$$p(x) : \text{Likelihood that } X_t \text{ first escapes } D = (0,5) \text{ to the left, i.e., becomes 0 or cancer-free}$$

**Figure:** Escape probability $p(x)$: (a) $a = 0.0, \varepsilon = 0.1$. (b) $a = 0.0, \varepsilon = 0.3$. 
Functional analytical methods: *Nonlocal PDEs*

- Escape probability
- Mean exit time
- Bifurcation
Mean exit time from a domain

\[ dX_t = f(X_t)dt + g(X_t)dL_t, \quad X_0 = x \]
How to quantify mean exit time?

\[
\begin{align*}
    dX_t &= f(X_t)dt + g(X_t)dL_t, \quad X_0 = x \in D \\
    f(\cdot) &\text{: Deterministic vector field}
\end{align*}
\]

**Exit time from a domain** \(D\): \(\sigma_x(\omega) = \inf\{t : X_t \in D^c\}\)

**Mean exit time** (for a ‘particle’ starting at \(x\)) from a domain \(D\):
\[
u(x) = \mathbb{E} \sigma_x(\omega)
\]

**Theorem:** Mean exit time \(u(x)\) satisfies
\[
    Au = -1 \text{ for } x \in D, \quad u|_{D^c} = 0,
\]
where \(A\) is the generator for this system.
One-dim system with $\alpha$–stable Lévy motion: $(0, d, \nu_\alpha)$

\[
dX_t = f(X_t)dt + dL_t, \quad X_0 = x \in D
\]

Generator:

\[
Au = f(x)u'(x) + \frac{d}{2}u''(x)
\]

\[
+ \int_{\mathbb{R}\setminus\{0\}} \left[ u(x + y) - u(x) - I_{\{|y|<1\}} yu'(x) \right] \nu_\alpha(dy)
\]

$\nu_\alpha(dx) = C_\alpha |x|^{-(1+\alpha)} dx$ with $C_\alpha = \frac{\alpha}{2^{1-\alpha} \sqrt{\pi}} \frac{\Gamma\left(\frac{1+\alpha}{2}\right)}{\Gamma\left(1 - \frac{\alpha}{2}\right)}$

$u(x)$: Mean exit time for a “particle” first escaping $D$

\[
Au = -1 \text{ for } x \in D, \quad u|_{D^c} = 0
\]
Example: Mean exit time for $\alpha$–stable Lévy motion

\[ dX_t = 0 \, dt + dL_t, \quad X_0 = x \]

**Getoor 1961**: Mean exit time from interval $D = (-r, r)$

\[ u(x) = \frac{\sqrt{\pi}}{2^\alpha \Gamma(1 + \frac{\alpha}{2}) \Gamma(\frac{1}{2} + \frac{\alpha}{2})} \left( r^2 - x^2 \right)^{\frac{\alpha}{2}} \]

**Figure**: Mean exit time from the interval ($-0.75, 0.75$)
Example: Mean exit time for Ornstein-Uhlenbeck system with $\alpha-$stable Lévy motion

\[ dX_t = -X_t \, dt + dL_t, \quad X_0 = x \]

$L_t$: Triplet $(0, 1, \nu_\alpha)$
Functional analytical methods: **Nonlocal PDEs!**

- Escape probability
- Mean exit time
- Bifurcation
Bifurcation under non-Gaussian noise

\[ dX_t = f(b, X_t)dt + dL_t \]

**Bifurcation**: How does this system evolve as parameters \( b, \alpha \) vary?

**Space of paths**: A big mess!

**Space of probability measures**: Order emerges out of orderless!
Bifurcation under non-Gaussian noise

\[ dX_t = f(b, X_t)dt + \epsilon dL_t. \]

Fokker-Planck equation for the stationary probability density function \( p(x) \):

\[ -[f(b, x)p(x)]' + \epsilon \int_{\mathbb{R}\setminus\{0\}} \left[ p(x + y) - p(x) - I\{|y|<1\}yp'(x) \right] \frac{dy}{|y|^{1+\alpha}} = 0 \]

under \( p(x) \geq 0, \int_{\mathbb{R}} p(x)dx = 1. \)

Chen, Duan & Zhang 2011
Example: Bifurcation under non-Gaussian noise

\[ dX_t = (bX_t - X_t^3)dt + dL_t \text{ for } b \in \mathbb{R}^1 \]

**Result:** Stationary prob. density function

- If \( b < 0 \) unimodal
- If \( b > 0 \) bimodal

**Figure:** \( \alpha = 1.99 \).

- (a) \( b = -30 \)
- (b) \( b = -50 \)
- (c) \( b = 20 \)
- (d) \( b = 30 \)
Conclusions

Nonlocal PDEs & Non-Gaussian Dynamics

- Escape probability
- Mean exit time
- Bifurcation

Computation!