**Question 1a:** Five people are sitting at a circular dinner table. In how many ways can they order chicken, fish, or beef such that nobody has the same dish as someone sitting next to them?

**Answer:** We use inclusion/exclusion.

Let the people at the table be numbered from 1 to 5 (clockwise), and let $c_i$ be the event that person $i$ has ordered the same dish as the person to their right.

Since there are five people and three dishes for each person to choose from, $N = S_0 = 3^5 = 243$.

If we assume 1 and the person sitting to his right order the same dish, we are reduced to four choices to make (as person 1’s dish is determined by that of his neighbor). Therefore $N(c_1) = 3^4$. The same holds true for each of the four other people, so $S_1 = \binom{5}{1}3^4 = 405$.

Next, we see that $n(c_1c_2) = 3^3 = 27$ (we choose dishes for the other three people, then persons 1 and 2 have their meal forced by their neighbors’ choices). The same holds for every pair, so $S_2 = \binom{5}{2}3^3 = 270$.

By the same argument, $S_3 = \binom{5}{3}3^2 = 90$ and $S_4 = \binom{5}{4}3^1 = 15$. However, $S_5$ breaks the pattern: Saying that all five events hold is equivalent to saying that all five people order the same dish, which can still happen in 3 ways.

Adding, $\mathcal{N} = S_0 - S_1 + S_2 - S_3 + S_4 - S_5 = 30$.

**Question 1b:** In how many ways can they order so that exactly two pairs of adjacent people share a dish?

**Answer:** We use generalized inclusion-exclusion with $m = 2$, which states that

$$E_2 = S_2 - \binom{3}{1}S_3 + \binom{4}{2}S_4 - \binom{5}{3}S_5 = 270 - (3)(90) + 6(15) - 10(3) = 60$$

**Question 1c:** In how many ways can the order so that exactly four pairs of adjacent people share a dish?

**Answer:** Using the same formula with $n = 4$ gives that

$$E_4 = S_4 - \binom{5}{1}S_5 = 15 - (5)(3) = 0$$

Alternatively, if four pairs of adjacent people share a dish, that’s everybody already, so the fifth pair does too.

**Question 2:** How many permutations of 1, 2, 3, 4, 5, 6, 7 are there such that none of the first three numbers are in the same position as before?
**Answer:** We use inclusion-exclusion as in the calculation of derangements. Let $c_i$ be the condition that integer $i$ is in position $i$.

We have $S_0 = 7!$, and $N(c_1) = N(c_2) = N(c_3) = 6!$, and $N(c_1c_2) = c(c_2c_3) = N(c_1c_3) = 5!$, and $N(c_1c_2c_3) = 4!$. Therefore the total number of such permutations is

$$7! - 3 \times 6! + 3 \times 5! - 4! = 3216$$

It would be fine not to simplify the left hand side.

**Question 3:** Compute the rook polynomial of the board below, where $O$ denotes a legal square for a rook.

```
O O O X
O O O X
O O O X
X X O O
```

**Answer:** Let $(*)$ be the leftmost $O$ in the bottom row. We have that $C_s =$

```
O O
O O
O O
```

and can see by inspection that $R(C_s, x) = 1 + 6x + 6x^2$. We also have $C_e =$

```
O O O X
O O O X
O O O X
X X X O
```

The rook polynomial of a $3 \times 3$ board is (again by inspection) $1 + 9x + 18x^2 + 6x^3$, so it follows that $R(C_e, x) = (1 + x)(1 + 9x + 18x^2 + 6x^3)$.

Using the relationship $r(C, x) = xr(C_s, x) + r(C_e, x)$, we therefore obtain

$$r(C, x) = x(1 + 6x + 6x^2) + (1 + x)(1 + 9x + 18x^2 + 6x^3)$$

**Question 4a:** Find the generating function for the sequence $0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \ldots$.

Your answer should be in closed form (no infinite sums)

**Answer:** This is equal to $x + x^2 + x^3 + x^4 + \ldots$, which is a geometric series with first term $x$, and ratio $\frac{1}{x}$. Using the standard formula, the sum is $\frac{x}{1 - x}$.

**Question 4b:** Find the generating function for the sequence $1, 1, 1, 2, 1, 1, 1, \ldots$ (where there is exactly one two and all other numbers are one)
Answer: This is $1 + x + x^2 + x^3 + 2x^4 + x^5 + x^6 + \cdots = (1 + x + x^2 + x^3 + x^4 + x^5 + \ldots) + x^4 = x^4 + \frac{1}{1-x}$.

Question 5a: Find the $x^{10}$ coefficient of $(x-3)^{-3}$. Do not leave negative binomial coefficients in your final answer.

Answer: By the binomial theorem, the term containing $x^{10}$ is

$$x^{10}(-3)^{-3-10} \binom{-3}{10} = x^{10}(-3)^{-13}(-1)^{10} \binom{12}{10}$$

$$= -\frac{12}{10} \binom{12}{10}$$

Question 5b: Find the $x^5$ coefficient of $\frac{x+2}{(x+1)^2}$. Do not leave negative binomial coefficients in your final answer.

Answer: We write $\frac{x+2}{(x+1)^2} = x(x+1)^{-2} + 2(x+1)^{-2}$.

The answer we are looking for is then the sum of the $x^4$ coefficient of $(x+1)^{-2}$ and twice the $x^5$ coefficient of $(x+1)^{-2}$, which is

$$\binom{-2}{4} + 2 \binom{-2}{5} = (-1)^4 \frac{5}{4} + 2(-1)^5 \frac{6}{5}$$

$$= 5 - 2 \times 6 = -7$$

Question 6: Find a generating function for the sequence of numbers $a_0, a_1, a_2, \ldots$ where $a_n$ is the number of partitions of $n$ into distinct parts, all of which are multiples of three.

Answer: This is a combination of two examples from class on Monday, and the answer is

$$f(x) = (1 + x^3)(1 + x^6)(1 + x^9)(1 + x^{12}) \ldots.$$ 

We use only $(1 + x^3)$ (instead of $(1 + x^3 + x^6 + x^9 + \ldots)$ because we want to use at most one of each part, and we only include the terms corresponding to multiples of three.