Math 2602 Spring 2009 Test 3

Each part is worth the listed number of points. **Show all work, and explain all your answers!**

1 (10 points). Using either Kruskal’s or Prim’s algorithm, find a minimum spanning tree of the graph below. Please make it clear in which order the edges are being added to your tree.

**Method 1:** We use Kruskal’s Algorithm. The cheapest edges are $CH$ and $CD$, which we add to our tree. The next cheapest are $DH$, $DE$, and $AB$. We reject $DH$ (since it forms the cycle $CDH$), add $DE$ and $AB$. Our next cheapest edges are $AF$, $EF$, and $AH$. We add $AF$ and $EF$, but reject $AH$ due to the cycle $AHDEF$. We next move to weight 4, where we add $FG$ to our tree. At this point we already have 7 edges, and we stop since we know a tree on 8 vertices has 7 edges.

**Method 2:** We use Prim’s algorithm, starting at $F$. We first add $AF$ (choosing arbitrarily from the two edges of weight 3 connecting $F$ to the outside world), then $AB$ (the cheapest edge connecting $A$ or $F$ to something else). We then add $AH$ (we could just as well have chosen $EF$), followed by $HC$, $CD$, and $DE$. Finally, we add $FG$.

**Note:** There are several possible different spanning trees you could have gotten (you have to make choices at several points in either algorithm, and the tree depends on those choices). However, all possible minimum trees have the same weight, 16.

2 (10 points). A truncated icosahedron, better known as a soccer ball, can be deflated into a planar graph with 12 pentagonal and 20 hexagonal regions. How many vertices and edges does this graph have?

**Answer:** The total number of sides of all the regions is $12 \times 5 + 20 \times 6 = 180$. This means there are \[ \frac{180}{2} = 90 \text{ edges.} \] Since there are 32 regions, plugging in $R = 32$ and $E = 90$ to Euler’s formula $V - E + F = 2$ gives $V = 60$. 


3 (5 points). What is the chromatic number of the complete bipartite graph $K_{4,4}$? Why?

**Answer:** The chromatic number is clearly at least 2 (two adjacent vertices have to be different colors), and you can get a 2 coloring just by coloring the left group of 4 vertices blue and the right group red (this works for any bipartite graph). Therefore the chromatic number is $\boxed{2}$.

4 (8 points). Find all solutions to the below system of equations:

\[
\begin{align*}
2w + x + 4y + z &= 7 \\
4w + 2x + 11y + 5z &= 21
\end{align*}
\]

**Answer:** In Matrix form this system is:

\[
\begin{pmatrix}
2 & 1 & 4 & 1 & 7 \\
4 & 2 & 11 & 5 & 21
\end{pmatrix}
\]

We start the row reduction by first multiplying row 1 by $\frac{1}{2}$, giving us:

\[
\begin{pmatrix}
1 & \frac{1}{2} & 2 & \frac{1}{2} & \frac{7}{2} \\
4 & 2 & 11 & 5 & 21
\end{pmatrix}
\]

then subtracting four times row 1 from row 2, leaving us with:

\[
\begin{pmatrix}
1 & \frac{1}{2} & 2 & \frac{1}{2} & \frac{7}{2} \\
0 & 0 & 3 & 3 & 7
\end{pmatrix}
\]

Finally we multiple row 2 by a third to leave us with the following matrix in Row Eschelon form:

\[
\begin{pmatrix}
1 & \frac{1}{2} & 2 & \frac{1}{2} & \frac{7}{3} \\
0 & 0 & 1 & 1 & \frac{7}{3}
\end{pmatrix}
\]

The columns corresponding to $x$ and $z$ don’t have any pivots, so we leave them alone (alternatively, we could write $[x = s]$ and $[z = t]$), and write everything else in terms of them. The second equation gives us:

\[
y = \frac{7}{3} - z = \frac{7}{3} - t = y
\]

and the first equation then gives us:

\[
w = \frac{7}{2} - \frac{z}{2} - 2y - \frac{x}{2} = \frac{7}{2} - t - \left(\frac{14}{3} - 2t\right) - \frac{s}{2} = \frac{7}{6} + \frac{3t}{2} - \frac{s}{2} = w
\]
5. A cafeteria serves either chicken or fish for dinner each evening. If it serves chicken on one night, then \(\frac{2}{3}\) of the time it will switch and serve fish the next night (otherwise serving chicken again). If it serves fish one night, then \(\frac{3}{4}\) of the time it will switch and serve chicken the next night (otherwise serving fish again).

a) (3 points) Write down a transition matrix for the Markov chain represented by this situation.

**Answer:** If the first row and column represents chicken and the second represents fish, the matrix is

\[
A = \begin{pmatrix}
\frac{1}{3} & \frac{2}{3} \\
\frac{3}{4} & \frac{1}{4}
\end{pmatrix}
\]

b) (3 points) If the cafeteria serves Chicken on Sunday, what is the probability it will serve chicken again on Tuesday?

**Method 1:** Since \((\begin{pmatrix} \frac{1}{3} \\ \frac{3}{4} \end{pmatrix})^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{11}{18} \\ \frac{7}{18} \end{pmatrix},\) the probability is \(\frac{11}{18}\).

**Method 2:** There is a \((\frac{1}{3})^2 = \frac{1}{9}\) probability that the cafeteria will serve chicken on Monday and on Tuesday. There is a \((\frac{2}{3})(\frac{2}{3}) = \frac{4}{9}\) probability it switches to fish on Monday, then switches back to chicken on Tuesday. Adding gives a total of \(\frac{11}{18}\).

**Note:** If you think about it, the two methods are really saying the same thing. All Method two was doing is describing in words how to find the upper left entry when you multiply the two matrices.

c) (8 points) Find all eigenvalues and eigenvectors of the matrix in part a.

**Answer:** The characteristic polynomial is

\[
\det(\lambda I - A) = \det \begin{pmatrix}
\lambda - \frac{1}{3} & -\frac{2}{3} \\
-\frac{3}{4} & \lambda - \frac{1}{4}
\end{pmatrix} = (\lambda - \frac{1}{3})(\lambda - \frac{1}{4}) - \frac{3}{4} \frac{2}{3} = \lambda^2 - \frac{7}{12} \lambda - \frac{5}{12}
\]

Setting this equal to 0 and solving gives that the eigenvalues are \(\lambda = 1, -\frac{5}{12}\) (Since \(A\) was a transition matrix, we knew in advance that 1 would be an eigenvalue).

For \(\lambda = 1\), we have \((I - A)v = \begin{pmatrix} \frac{2}{3} \\ \frac{3}{4} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0\). Solving this gives \(r \begin{pmatrix} 9 \\ 8 \end{pmatrix}\) as an eigenvector.

For \(\lambda = -\frac{5}{12}\), we have \((I - A)v = \begin{pmatrix} -\frac{3}{4} \\ \frac{3}{4} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0\). Solving this gives \(r \begin{pmatrix} -1 \\ 1 \end{pmatrix}\)

d) (3 points) In the long run, on what percentage of days will the cafeteria serve fish?

**Answer:** We take the \(\lambda = 1\) eigenvector and find a multiple adding to 1. Solving \(9r + 8r = 1\) gives \(r = \frac{1}{17}\). Plugging into our eigenvector gives \(\begin{pmatrix} \frac{9}{17} \\ \frac{8}{17} \end{pmatrix}\), so the fish is served on \(\frac{8}{17} \approx 47\%\) of the days.