1. Solutions to Professor Andrew's Sample Problems

1. Both graphs are planar, so you just need to give a planar drawing.

2. The chromatic number of $C_n$ is 2 if $n$ is even, as you can color the vertices alternating red, blue. However, you cannot color $C_n$ with only 2 colors if $n$ is odd (your only hope would be to alternate between the colors, but if you do that the first and last vertices in the cycle are the same color). However, it is easy to color $C_n$ with 3 colors, so the chromatic number is 3 when $n$ is odd.

For $W_n$, the central vertex must be a different color than every other vertex in the cycle. It follows that the chromatic number of $W_n$ is one more than that of $C_n$, meaning it is 3 for even $n$ and 4 for odd $n$.

3. For the left graph, an example of a 3 coloring would be if $a$ and $e$ were red, $b$ and $d$ were yellow, and $c$ and $f$ were blue. Since the graph has a triangle, no 2 coloring is possible and the chromatic number is 3. Similarly, the chromatic number of the right graph is 3 since it contains a triangle and we see directly that coloring $a$ and $b$ red, $c$ and $d$ yellow, and $e$ and $f$ blue works.

The chromatic number of any tree on at least 2 vertices is 2. We can prove this by induction. The base case (2 vertex trees) is clear. Now let $T$ be any tree, and $v$ be a vertex of degree one (we know any tree has at least two such vertices). We first color everything except for $v$ by the inductive hypothesis, then color $v$ a different color than its neighbor.

Note that this is the same argument we did in class to show that planar graphs were 6 colorable.

4. Left: $(x, y, z) = (5 - 3r, -2 + r, r)$ for any real $r$. Middle: No solution. Right: $(x, y, z) = (\frac{1}{3}, 2, -\frac{1}{3})$.

5.

6. Note that $P^{-1}AP = D$ is the same thing as $A = PDP^{-1}$, so the question is just asking about diagonalization. We know from the given polynomial that the eigenvalues are $\{-1, 1, 2\}$. We next find the eigenvectors by solving $(\lambda I - A)\mathbf{v} = 0$. Doing so gives eigenvectors of $(2, 1, 0)^T$ (for $\lambda = -1$), $(1, 1, 1)^T$ (for $\lambda = 1$), and $(1, 1, 2)^T$ (for $\lambda = 2$). This gives an answer of

$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$.
Note: Your columns may appear in a different order in your $P$ than in mine. This is fine as long as the same switches also happen in the entries of $D$.

7. As in the Fibonacci example from class, we let $b_n = a_{n+1}$, so the recurrence becomes

\[ b_{n+1} = 2b_n - 3a_n \]

\[ a_{n+1} = b_n, \]

Or in other words

\[
\begin{pmatrix}
  b_{n+1} \\
  a_{n+1}
\end{pmatrix} =
\begin{pmatrix}
  2 & 3 \\
  1 & 0
\end{pmatrix}
\begin{pmatrix}
  b_n \\
  a_n
\end{pmatrix}
\]

We next find the eigenvalues/vectors of the matrix in the recurrence, finding that its eigenvalues are 3 (with eigenvector $(3,1)^T$) and $-1$ (with eigenvector $(-1,1)^T$). At this point we have two options:

**Method 1:** Our initial conditions were $b_0 = -1$, $a_0 = 7$. Since

\[
\begin{pmatrix}
  -1 \\
  7
\end{pmatrix} = \frac{3}{2} \begin{pmatrix}
  3 \\
  1
\end{pmatrix} + \frac{11}{2} \begin{pmatrix}
  -1 \\
  1
\end{pmatrix},
\]

we have

\[
\begin{pmatrix}
  b_n \\
  a_n
\end{pmatrix} = \frac{3}{2} 3^n \begin{pmatrix}
  3 \\
  1
\end{pmatrix} + \frac{11}{2} (-1)^n \begin{pmatrix}
  -1 \\
  1
\end{pmatrix}.
\]

Comparing the bottom entries, we see $a_n = \frac{3}{2} 3^n + \frac{11}{2} (-1)^n$.

**Method 2:** We diagonalize as in previous problem, writing

\[
\begin{pmatrix}
  2 & 3 \\
  1 & 0
\end{pmatrix} = \begin{pmatrix}
  3 & 1 \\
  -1 & 1
\end{pmatrix} \begin{pmatrix}
  3 & 0 \\
  0 & -1
\end{pmatrix} \begin{pmatrix}
  3 & 1 \\
  -1 & 1
\end{pmatrix}^{-1}.
\]

We then have

\[
\begin{pmatrix}
  2 & 3 \\
  1 & 0
\end{pmatrix}^n = \begin{pmatrix}
  3 & 1 \\
  -1 & 1
\end{pmatrix} \begin{pmatrix}
  3^n & 0 \\
  0 & (-1)^n
\end{pmatrix} \begin{pmatrix}
  3 & 1 \\
  -1 & 1
\end{pmatrix}^{-1}.
\]

We then multiply this out and multiply by $(-1,7)^T$, simplifying to get the same answer. **Note: I would consider this problem too messy computationally to stick on a one hour exam.**

8. a) The transition matrix is $A = \begin{pmatrix}
  0.05 & 0.45 \\
  0.95 & 0.55
\end{pmatrix}$.

b) Since $A \begin{pmatrix}
  0.8 \\
  0.2
\end{pmatrix} = \begin{pmatrix}
  0.13 \\
  0.87
\end{pmatrix}$, we would expect 87 percent to be sick tomorrow. A similar calculation with $A^2$ gives that we expect about 60.2 percent to be sick the day after tomorrow.
c) The solution to \((I - A)v = 0\) is \((\frac{9r}{19}, r)\). Remembering that a probability vector has to add to 1, we see that \(r = \frac{19}{26}\). Plugging in, we see that in the long run we expect \(\frac{19}{28}\) of the population to be sick.

9) Recall that the columns of a transition matrix add up to 1. This immediately gives that the missing entries are (from left to right) 0.1, 0.8, and 0.7. Similar calculation as in the previous problem gives a steady state of \((\frac{1}{4}, \frac{1}{2}, \frac{1}{4})^T\).

2. Solutions to Exam 3 Review Sheet

2.1. Graph Theory. 2. Since every edge is contained in two regions, and there are 20 triangles, we have \(3(20) = 2E\), so \(E = 30\). Since \(V - E + R = 2\), plugging in \(R = 20\) gives \(V = 12\).

3. The graph is not planar, as the relabelling below shows:

4. Many examples are possible, including the graph in the previous problem (which you should be able to show has chromatic number 3).

5. See Professor Andrew’s Sample number 2.

2.2. Linear Algebra.

\[
\begin{bmatrix}
1 & 2 & 1 & 7 \\
1 & -2 & 3 & 12 \\
2 & 0 & u & q
\end{bmatrix} \Rightarrow \begin{bmatrix}
1 & 2 & 1 & 7 \\
0 & -4 & 2 & 15 \\
0 & -4 & 2 & 14
\end{bmatrix}
\]

\[a - 19 = 0 \Rightarrow a = 19\]

\[Z\] is non-pivot

\[y = -\frac{5}{4} + \frac{1}{2}Z\]

\[x = 7 - Z - 2y = 7 - Z - (-\frac{5}{2} + Z) = \frac{19}{a} - 2Z\]
2: \[
\begin{bmatrix}
11 & 2 & 3 \\
11 & 2 & 2 \\
22 & 1 & 3
\end{bmatrix} \Rightarrow \begin{bmatrix}
11 & 2 & 3 \\
0 & 0 & -1 \\
0 & 0 & 3
\end{bmatrix} \Rightarrow \begin{bmatrix}
11 & 2 & 3 \\
0 & 0 & -1 \\
0 & 0 & -3
\end{bmatrix} \Rightarrow \begin{bmatrix}
11 & 2 & 3 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix} \leftarrow \text{Row Echelon}
\]

\[
\begin{bmatrix}
11 & 2 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{bmatrix} \Rightarrow \begin{bmatrix}
11 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix} \leftarrow \text{Reduced Row Echelon}
\]

3a: \[
\lambda I - A = \begin{bmatrix}
\lambda - 1 & -1 \\
-1 & \lambda - 1
\end{bmatrix} = (\lambda - 1)^2 - (-1)^2 = \lambda^2 - 2\lambda - 15 = 0
\]
\[\lambda = 5, 3\]

\[\lambda = 5 \quad \begin{bmatrix}
-1 & -1 \\
-1 & 5
\end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0, \quad \begin{bmatrix} x \\ y \end{bmatrix} = r \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]
\[\lambda = 3 \quad \begin{bmatrix}
-1 & -1 \\
-1 & 3
\end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0, \quad \begin{bmatrix} x \\ y \end{bmatrix} = r \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\]

1) Since \[
\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2} \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)
\]
\[
\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \frac{1}{2} \left( 5^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 3^n \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right) \Rightarrow \begin{cases} x_n = \frac{5^n + 3^n}{2} \\ y_n = \frac{5^n - 3^n}{2} \end{cases}
\]

4) Markov Chain.
Transition matrix
\[
\begin{bmatrix}
\frac{1}{2} & 1 \\
\frac{1}{2} & 0
\end{bmatrix}
\]
\[
\lambda = 1, \quad \begin{bmatrix}
\frac{1}{2} & -1 \\
-\frac{1}{2} & 1
\end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0
\]
\[x = 2r, \quad y = r
\]
\[x + y = 1 \quad \text{so} \quad r = \frac{1}{3}
\]
\[
\begin{cases} x = \frac{2}{3} \leftarrow \text{Sunny} \\ y = \frac{1}{3} \leftarrow \text{Rainy} \end{cases}
\]