1. Graph Theory

The graph theory covered on test 3 corresponds to sections 12.3, 13.1, and 13.2. Among the subjects/topics you should know are:

- How to use at least one of Kruskal’s and Prim’s algorithms to find a minimum spanning tree of a graph, and to give a step by step description of how you found the tree. Which one you use is a matter of your own preference.
- Euler’s formula for planar graphs, along with its implications in terms of counting the number of regions for a planar graph
- How to show a graph either is or is not planar. To show a graph is planar, the easiest way is just to exhibit a drawing without crossing edges. To show it is not planar, you should either show it has too many edges (e.g. more than $3V - 6$ edges) or show a way to start with either $K_5$ or $K_{3,3}$ and add edges/vertices to get the graph in question.
- The definition of chromatic number, and how to find it for a graph. Remember that to show that a graph has chromatic number $k$, you need to show both that it CAN be colored with $k$ colors (e.g. by giving an explicit coloring) and that it CANNOT be colored with $k - 1$ colors.
- The statement of the 4 color theorem

Sample problems would include

1. For the graph labelled (d) on page 391, find a minimum spanning tree using either Kruskal’s or Prim’s algorithm. Be sure to indicate which edges get added to your tree in which order.

2. Let $G$ be a connected planar graph which divides the plane into 20 regions, every single one of which is a triangle. How many edges does $G$ have? How many vertices does $G$ have?

3. Is the below graph planar? Why or why not?

4. Give an example of a non-planar graph with chromatic number less than 4.

5. The wheel graph $W_k$ is formed by first taking a cycle of $k$ vertices, then adding a $k+1^{st}$ vertex in the center connected to every vertex on the outside. Find the chromatic number of $W_k$ (your answer will depend on $k$).
2. Linear Algebra

The linear algebra covered on test 3 corresponds to chapters 2 and 7 from part III of your text, as well as the additional material on recurrence relations and Markov Chains. Among the subjects/topics you should know are:

- How to use row reduction to put a matrix in Row Echelon (or Reduced Row Echelon) form.
- Using row reduction to find all solutions of a system of equations, as well as how to show that no solution exists at all.
- The definition of Eigenvalues and Eigenvectors, as well as some of their basic properties. You should also be able to find them given a $2 \times 2$ or $3 \times 3$ matrix.
- How to use diagonalization to write a matrix as $A = PDP^{-1}$, as well as its implications in terms of finding $A^n$.
- How to convert a system of linear recurrences into a single matrix recurrence, and how to use diagonalization (or eigenvectors) to solve that recurrence.
- How to set up a Markov Chain to describe a situation, and to find the steady state solutions exactly.
- How to approximate the long term behavior system with multiple “absorbing” states using a calculator.

Sample problems would include

1. Find all $a$ for which the system of equations

\[
\begin{align*}
    x + 2y + z &= 7 \\
    x - 2y + 3z &= 12 \\
    2x + 4z &= a
\end{align*}
\]

Has at least one solution. For each such $a$, find all solutions.

2. Put the matrix

\[
\begin{bmatrix}
    1 & 1 & 2 & 3 \\
    1 & 1 & 2 & 2 \\
    2 & 2 & 1 & 3
\end{bmatrix}
\]

into reduced row echelon form.

3a. Find the eigenvalues and eigenvectors of the following matrix:

\[
\begin{bmatrix}
    1 & 4 \\
    4 & 1
\end{bmatrix}
\]

3b. Solve the following recurrence:

\[
\begin{align*}
    x_{n+1} &= x_n + 4y_n \\
    y_{n+1} &= 4x_n + y_n \\
    x_0 &= 1 \\
    y_0 &= 0
\end{align*}
\]
4: In the land of Nod it either rains or is sunny on each day (and never more than one of these on the same day). It is never rains two days in a row, and if a day is sunny it is equally likely to be followed by rain or sun the next day. In the long run, what proportion of days are sunny in Nod?