Math 2602 Spring 2009 Test 1

Each part is worth the listed number of points. **Show all work, and explain all your answers!**

1 (10 points) Prove by the principle of mathematical induction: \(1 + 5 + 9 + \cdots + (4n - 3) = 2n^2 - n\)

**Answer:** The base case is \(n = 1\), where we can directly check that 1 = 2(1)^2 – 1.

For the inductive step, assume that \(1 + 5 + 9 + \cdots + (4n - 3) = 2n^2 - n\). Then

\[
1 + 5 + 9 + \cdots + (4(n + 1) - 3) = [1 + 5 + \cdots + (4n - 3)] + (4n + 1)
\]

\[
= (2n^2 - n) + 4n + 1
\]

\[
= 2(n^2 + 2n + 1) - (n + 1) = 2(n + 1)^2 - (n + 1)
\]

where the second inequality came from our inductive hypothesis.

2 (5 points for each part): A restaurant employs a total of 12 different chefs, of which 9 are trainees and three are “master” chefs. To prepare a four course meal, exactly one chef needs to prepare each course.

a) In how many ways can this be done, assuming that no chef can work on more than one course? (it matters who prepares which course).

**Answer:** There are 12 choices for who does the first course, 11 for who does the second, 10 for who does the third, and 9 for who does the fourth. By the rule of product, this is \((12)(11)(10)(9) = 11880\).

Alternatively, you could have thought of this as \(P(12, 4)\).

b) In how many ways can this be done if we add the further restriction that of the four chefs preparing the meal, at least one must be a master chef?

**Answer:** There are \((9)(8)(7)(6) = 3024\) choices for the four chefs which have **no** master chefs. We subtract these off, and we are left with \(11880 - 3024 = 8856\) choices which have at least one master chef.
3 (5 points each): For each of the following two statements, determine whether it is true or false. Explain your answers: (i): $n^2 = O(e^n)$. (ii): $e^n = O(n^2)$.

**Answer:** $\lim_{n \to \infty} \frac{n^2}{e^n} = \lim_{n \to \infty} \frac{2n}{e^n} = 0$ by L'Hopital's rule. Since this limit is finite, $n^2 = O(e^n)$. But the same method gives $\lim_{n \to \infty} \frac{e^n}{n^2} = \infty$, so $e^n \neq O(n^2)$.

You could also have argued some other way that exponentials grow much faster than quadratics. What I'm more concerned with is your explanation conveyed the idea that $n^2$ grew significantly slower than $e^n$, and what this meant in terms of $O$ notation.

4 (10 points): Find, BUT DO NOT SOLVE, a recurrence relation (including initial conditions) for the number of ways of tiling a $1 \times n$ strip with $1 \times 2$ and $1 \times 3$ dominoes. Again, you only need to set up the recurrence.

**Answer:** Let $a_n$ be the number of ways of tiling a $1 \times n$ strip. We can classify our tilings based solely on what kind of domino is used to tile the very last square on our strip. There are two cases:

Case 1: The last tile is a $1 \times 2$ domino. We still have to tile the remaining $1 \times (n - 2)$ strip, so there are $a_{n-2}$ ways to tile the remainder.

Case 2: The last tile is a $1 \times 3$ domino. We still have to tile the remaining $1 \times (n - 3)$ strip, so there are $a_{n-3}$ ways to tile the remainder.

Adding, we see that $a_n = a_{n-2} + a_{n-3}$. We need three initial conditions because our recurrence stretches back from $a_n$ to $a_{n-3}$. Just by inspection we can see that $a_1 = 0$, $a_2 = a_3 = 1$. (both tiles are too large for a length 1 strip, and a length 2 or 3 can only be covered by exactly 1 domino.

5 (10 points): Solve the following recursion: $a_n − 4a_{n−1} + 4a_{n−2} = 4^n$, $a_0 = 0$, $a_1 = 0$.

We first solve $a_n − 4a_{n−1} + 4a_{n−2} = 0$. This recurrence has characteristic polynomial $x^2 − 4x + 4 = (x−2)^2$, so the general solution is $a_n = (An + B)2^n$.

We next look for a specific solution. Since the right hand side is $4^n$, we guess $a_n = C(4^n)$. Plugging in, we see that our guess will only work if

$$4^n = a_n - 4a_{n-1} + 4a_{n-2} = C(4^n) - 4C(4^{n-1}) + 4C4^{n-2} = C(4^n) - C(4^n) + \frac{C}{4}4^n = C \frac{4}{4}4^n$$

Comparing, we see that $C = 4$ is a solution.

So our solution to the original recurrence (w/o initial conditions) is $(An + B)2^n + 4(4^n)$. We now plug in initial conditions to get

$$0 = B + 4(\text{from plugging in } n = 0)$$

$$0 = (A + B)2 + 16(\text{from plugging in } n = 1)$$

Solving, $B = -4$, $A = -4$, so our final solution is $a_n = 4(4^n) − (4n + 4)2^n$.  

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