Recall the Principle of Inclusion-Exclusion (or one version of it, at least):

**Proposition 0.1.** Let $S$ be a set of objects, and let $c_1, \ldots, c_r$ be a list of conditions. Let $N$ be the number of objects, and let $N(c_1)$ denote the number of objects satisfying condition 1, $N(c_1 c_4)$ be the number of objects satisfying both $c_1$ and $c_4$, etc. We also let

$$
S_0 = N = |S| \\
S_1 = N(c_1) + N(c_2) + \cdots + N(c_r) \\
S_2 = N(c_1 c_2) + N(c_1 c_3) + \cdots + N(c_r-1 c_r) \\
\vdots \\
S_r = N(c_1 c_2 \cdots c_r)
$$

Then the number of objects satisfying **none** of the conditions is $S_0 - S_1 + S_2 - S_3 \cdots + (-1)^r S_r$.

The sum in $S_1$ is all of the objects taken 1 at a time, the sum in $S_2$ is all of the objects taken 2 at a time (meaning it contains $\binom{r}{2}$ terms), and so on. This is really just a restatement of Principle 6.1.2 from your book, which can be thought of as the following corollary.

**Corollary 0.2.** Let the $S_i$ be as above. Then the number of objects satisfying **at least one** of the conditions is $N - \overline{N} = S_0 - S_1 + S_2 - S_3 \cdots - (-1)^r S_r$.

There are really two stumbling blocks you’ll typically run across with the principle. The first is recognizing when you need to use it, that is when the (much simpler) counting techniques we’ve used before fail. The second is actually figuring out what all of the $S_i$ are. So let’s take a look at some examples where we can see what’s going on.

**Example 1:** A poker game is played with a deck of 52 cards, including 13 spades, 13 hearts, 13 diamonds, and 13 clubs. Each player is dealt a hand of 5 cards from the deck. What are the odds that your hand of 5 cards is missing at least one suit?

The phrase “what are the odds” suggests to me that this is a probability problem, meaning we should be looking to count the total number of hands missing a suit, then divide it by the total number of hands. The latter is not hard to compute. There are $\binom{52}{5}$ ways to choose 5 cards from the deck.

It’s also not hard to figure out how many hands are missing any particular suit. For example, there are 39 cards in the deck which are **not** clubs, so there are $\binom{39}{5}$ ways to choose 5 non-clubs for our hand. What’s tempting to do, but **WRONG**, is to just add up: There are 4 suits we could be missing, so we would say there are $4 \binom{39}{5}$ ways to be missing a suit.

The problem is that in doing it that way we’re overcounting. A hand of 5 spades is being counted as “hand without clubs”, “hand without diamonds”, and “hand without spades”, when we want to only count it once. So we instead use inclusion exclusion.

Our 4 conditions are

- $c_1$: The hand contains no clubs
- $c_2$: The hand contains no diamonds
- $c_3$: The hand contains no hearts
• $c_4$: The hand contains no spades

As above, $N = S_0 = \binom{52}{5} = 2598960$.

Since $N(c_1) = N(c_2) = N(c_3) = N(c_4) = \binom{39}{5}$, $S_1 = 4\binom{39}{5}$

For each set of two suits we could be missing, there are $\binom{52}{5}$ ways to choose 5 cards not in those suits. It follows that $N(c_1c_2) = N(c_1c_3) = N(c_1c_4) = N(c_2c_3) = N(c_2c_4) = N(c_3c_4) = \binom{26}{5}$.

Adding up, $S = 6\binom{26}{5}$.

Similarly, $N(c_1c_2c_3) = N(c_1c_2c_4) = N(c_1c_3c_4) = N(c_2c_3c_4) = \binom{13}{5}$, so $S_4 = 4\binom{13}{5}$.

$N(c_1c_2c_3c_4) = 0$, since every hand has to have at least one card.

We’re looking for how many hands are missing a suit, that is how many hands DO satisfy a condition, so our formula is $4\binom{39}{5} - 6\binom{26}{5} + 4\binom{13}{5} = 1913496$.

Our final answer is therefore $\frac{1913496}{2598960} = 0.736$.

What can we take away from this example?

• The point when I knew I was going to have to use inclusion-exclusion was where the naive way led to overcounting. This is a good general rule: If you overcount, and can’t see a quick way to correct for it, inclusion-exclusion may provide the answer.
• There was a lot of symmetry hiding inside in the counting; once I figured out how many hands were missing one pair of suits, I knew it was the same for all of them. In practice, I could have just multiplied by 4, 6, and 4 (which came from $\binom{4}{1}, \binom{4}{2}$, and $\binom{4}{3}$) without worrying about listing out all of the $N(c_2c_4)$ and so forth.
• This problem was long. Inclusion exclusion is like that at times, unfortunately.

Example 2: I have 9 pieces of candy, each of a different brand. I want to divide the candy up among 4 children in such a way that every child gets at least one piece. In how many ways can this be done?

If we didn’t have this extra condition that everyone get a piece, the answer would have been $4^9$ (for each piece of candy we have 4 choices which child gets it). But now we have this extra condition, which we can think of as really being 4 different conditions.

• $c_1$: Child A gets at least one piece
• $c_2$: Child B gets at least one piece
• $c_3$: Child C gets at least one piece
• $c_4$: Child D gets at least one piece

As mentioned above, $N = S_0 = 4^9$.

For each child, there are $3^9$ ways to avoid giving that child a piece of candy, so $S_1 = \binom{4}{1}3^9 = 4\times 3^9$. 
For each pair of children, there are \(2^9\) ways to avoid giving candy to both children in the pair, so
\[ S_2 = \binom{4}{2}2^9. \]

Similarly, \(S_3 = \binom{4}{3}1^9\) and \(S_4 = \binom{4}{4}0^9 = 0\) (we’re giving out all the candy, so it’s impossible for everybody to end up with no pieces).

We want NONE of the conditions to be satisfied, so we take
\[ S_0 - S_1 + S_2 - S_3 + S_4 = 4^9 - 4 \times 3^9 + 6 \times 2^9 - 4 \times 1^9 = 186,480. \]

**Question 0.3.** What happens if I’m allowed to keep some of the candy for myself, but still require that every child gets a piece?

**Example 3:** A lazy coat-checker collects 6 different coats from 6 different people. At the end of the evening, he returns one coat to each person without checking who gets what coat. As it so happens, nobody gets their own coat back. In how many ways can this happen, and how likely is it?

The objects we’re looking at here are all of the different ways he can return 6 coats to 6 people, of which there are \(6! = P(6,6) = 720.\) The conditions we’re looking to avoid can be written as \(c_i: \text{Person } i \text{ gets their own coat back}.\) We will use inclusion-exclusion again.

As noted, \(S_0 = 6! = 720.\)

Condition \(c_1\) corresponds to person 1 getting their own coat back. Once we assume that this happens, there are 5! ways to give the coats back to the remaining 5 people. In other words, \(N(c_1) = 5! = 120.\) Since there are 6 different choices for a single condition, \(S_1 = 6(5!) = 720.\)

A condition like \(c_1c_2\) corresponds to two people getting their own coat back, and the remaining 4 coats being given out willy-nilly among the remaining 4 people. There are 4! ways to give out those other 4 coats, so \(N(c_1c_2) = 4!.\) There are \(\binom{6}{2}\) choices of two conditions (two people), so
\[ S_2 = \binom{6}{2}4! = \frac{6!}{2!4!} = \frac{6!}{2!} \]

We could keep on going like this, but let’s skip ahead to the general case. If we’re looking for \(S_k,\) we’re looking at lists of \(k\) conditions that must be satisfied. There are \(k\) people who get their own coat, so \(6 - k\) people who are left and \(n - k\) coats to give out. These coats can be given out in \((6 - k)!\) ways. However, we still need to multiply by number of different sets of \(k\) conditions to be satisfied. This gives
\[ S_k = \binom{6}{k}(n - k)! = \frac{6!}{k!(6 - k)!}(6 - k)! = \frac{6!}{k!} \]

Adding up, we see that \(\overline{N} = S_0 - S_1 + S_2 - \cdots = \frac{6!}{0!} - \frac{6!}{1!} + \frac{6!}{2!} - \cdots + \frac{6!}{6!} = 265\)

**Definition 0.4.** A Derangement of \(n\) objects is a permutation so that none of the objects are in the correct position.

The exact same argument as above gives that \(D_n,\) the number of arrangements of \(n\) objects, satisfies
\[ D_n = n!\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right)\]
Interesting things to note here:

- Since \( n! \) grows so quickly, the terms in our formula go to zero very quickly as well.
- We know from calculus that \( e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \). Plugging in \( x = -1 \), we see that

\[
\frac{1}{e} = \left( \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} \cdots + \frac{(-1)^n}{n!} + \ldots \right).
\]

The terms we leave out in truncating the sum at \( n \) are, as noted above, very small. So a good approximation is to say that \( D_n \approx \frac{n!}{e} \). For example, \( \frac{6!}{e} = 264.873 \). Because the Taylor series is alternating, we know the error in this approximation (the difference between truncating the Taylor series after \( n \) terms and letting it go on to infinity) is at most \( \frac{1}{(n+1)!} \). The error in this approximation is at most \( \frac{1}{(n+1)!} \). Since \( D_n \) better be an integer, you can just round the approximation, and you’ll have the correct answer!

- How likely is it nobody gets their own coat back? The exact probability is \( \frac{D_n}{n!} \), but a good approximation is \( \frac{n!}{en} = \frac{1}{e} \).

- A different way we could have seen this approximation is to note that each person has a \((1 - \frac{1}{n})\) chance of getting their own coat back. If these events were independent, the odds nobody got their coat back would be \((1 - \frac{1}{n})^n \approx \frac{1}{e}\) for large \( n \). The events are NOT independent (if I don’t get my own coat back, it’s slightly more likely that I get your coat, and therefore less likely that you get your coat), but it’s an okay approximation if there are many people.

What if we instead asked what the odds were that exactly \( k \) people got their own coat back? There are \( \binom{n}{k} \) to choose the \( k \) people to get their own coat back, and \( D_{n-k} \) ways for none of the remaining \( n-k \) people to get their own coat back. The odds are therefore

\[
\binom{n}{k} \frac{D_{n-k}}{n!} \approx \frac{n! \binom{n}{k} \frac{(n-k)!}{e}}{n!} = \frac{1}{k! e}
\]