Recall from last time that we had a multisymplectic manifold (X, ω); ω is a multisymplectic 3-form.

Also, we had a set of 1-forms $Ham(X, \omega)$:

$\alpha \in Ham(X, \omega) \implies \exists\, v_\alpha \in Vect(X)$ s.t.

$d\alpha = -\omega(v_\alpha, -, -)$. Lastly, we had two bracket operations

$$\{\alpha, \beta\} := \mathrm{Lie}_{v_\alpha} \beta$$

&

$$\{\alpha, \beta\}' := \omega(v_\alpha, v_\beta, -).$$

We showed how $(Ham(X), \{\cdot,\cdot\})$ & $(Ham(X), \{\cdot,\cdot\}')$ are weak Lie 2-algebras, i.e. a category with

- objects as elements of $Ham(X)$
- morphisms in $Ham(X) \oplus C^\infty(X)$

$$\alpha \xmapsto{(\alpha, f)} \alpha + df$$

- $\{\cdot,\cdot\}$ (resp. $\{\cdot,\cdot\}'$) is a bilinear functor

- a natural transformation

$$S : \{\cdot,\cdot\} \Longrightarrow -\{\cdot,\cdot\} * \sigma \qquad \text{(alternator)}$$

(resp. for $\{\cdot,\cdot\}'$)

- a natural transformation

$$J : \{\cdot,\{\cdot,\cdot\}\} \Longrightarrow \{\{\cdot,\cdot\},\cdot\} + \{\cdot,\{\cdot,\cdot\}\} * \sigma_{12}$$

(Jacobiator)

For $(\mathrm{Ham}(X), \{\cdot,\cdot\})$, we have nontrivial S but $J = 1$.

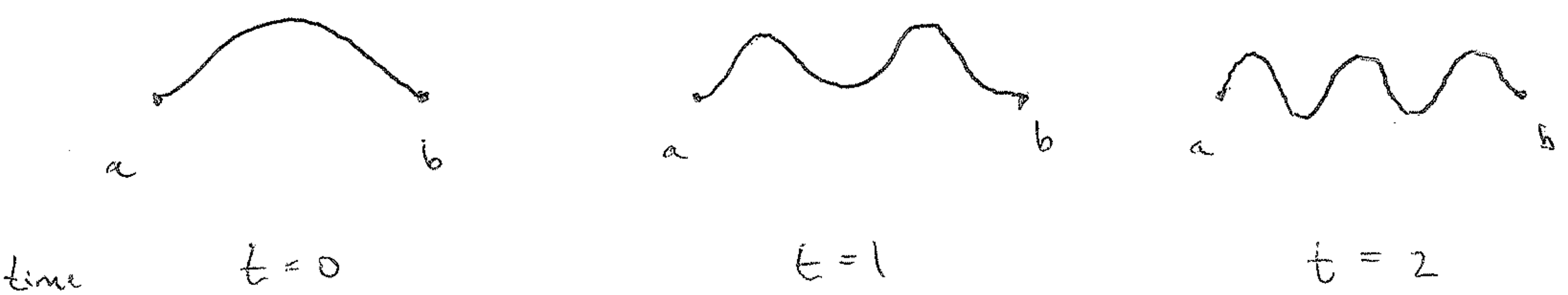
For $(\mathrm{Ham}(X), \{\cdot,\cdot\}')$, we have trivial S, but non-trivial J.

<u>Theorem</u>: (Baez, Hoffnung, Rogers)

$(\mathrm{Ham}(X), \{\cdot,\cdot\})$ & $(\mathrm{Ham}(X), \{\cdot,\cdot\}')$ are isomorphic as weak Lie 2-algebras.

One of our motivations is to develop a geometric picture of classical string dynamics in analogy with classical particles & symplectic geometry.

<u>Example</u>: A non-relativistic (classical) string with 1 transverse degree of freedom (i.e. wiggles in only 1 direction.)



To describe the configurations of the string, we define a function

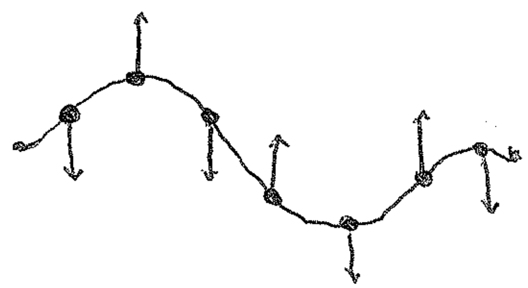
$$\phi : \mathbb{R} \times [a,b] \longrightarrow \mathbb{R}$$

$\phi(\tau, \sigma)$ is the amplitude of the string at time τ & position σ.

At each time τ, we can make measurements of things like energy, "wave momentum", etc.

To figure out the formula for the energy: pretend the string is composed of "beads":

figure out the kinetic energy:



Each bead is a classical particle:

each bead will have kinetic energy $\frac{1}{2} m v_i^2$

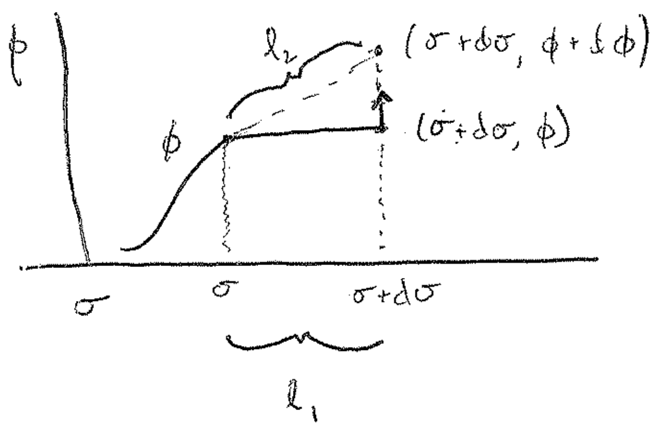
with $v_i = \dfrac{\partial \phi(\tau, \sigma_i)}{\partial \tau}$

So we have total kinetic energy

$$T = \sum_{i=1}^{N} \frac{1}{2} m \left(\frac{\partial \phi}{\partial \tau}\right)^2 = \frac{1}{2} \int_a^b \left(\frac{\partial \phi}{\partial \tau}\right)^2 d\sigma$$

(take $m = 1$)

For potential energy:



$$\ell_1 = d\sigma$$

$$\ell_2 = \left((d\sigma)^2 + (d\phi)^2 \right)^{1/2}$$

$$\Delta \ell = \ell_2 - \ell_1$$

$$= d\sigma \left[\left(1 + \left(\frac{\partial \phi}{\partial \sigma} \right)^2 \right)^{1/2} - 1 \right]$$

Now we Taylor expand $x^{1/2}$:

$$\Delta \ell \approx d\sigma \left[1 + \frac{1}{2} \left(\frac{\partial \phi}{\partial \sigma} \right)^2 - 1 + \dots \right]$$

$$\approx d\sigma \, \frac{1}{2} \left(\frac{\partial \phi}{\partial \sigma} \right)^2$$

So

$$V = \int_a^b \frac{1}{2} \left(\frac{\partial \phi}{\partial \sigma} \right)^2 d\sigma$$

Therefore the energy of the string at time τ is

$$\mathcal{H} = \frac{1}{2} \int_a^b \left[\left(\frac{\partial \phi}{\partial \tau} \right)^2 + \left(\frac{\partial \phi}{\partial \sigma} \right)^2 \right] d\sigma$$

Allowed configurations are given by the wave equation

$$\frac{\partial^2 \phi}{\partial \tau^2} - \frac{\partial^2 \phi}{\partial \sigma^2} = 0$$

(plus boundary data.)

Possible geometric pictures to describe string dynamics.

(1) Let M be the space of all possible string configurations.

(i.e. maps from $[a, b]$ to $\mathbb{R}$)

We could then describe dynamics as a path

$$\gamma : \mathbb{R} \longrightarrow M$$

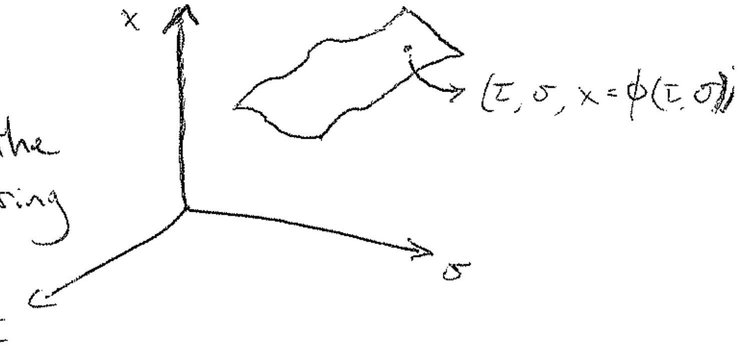
where $\gamma(t)$ is the configuration space of the string at time t.

Problem: $\dim(M) = \infty$

If you want to quantize the string, M is a bad place to do it.

(2) "A space-time picture"

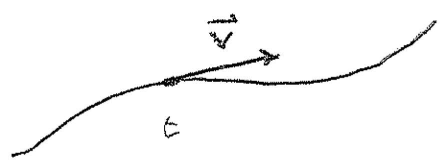
worldsheet of the string



So solutions to the wave equation are submanifolds.

- This is called the extended configuration space.

- Every particle has a velocity at a point on its worldline:



- Every point on the string world sheet has a "velocity" bivector.

The phase space for the string:

at every point we get momentum data attached

$$p_\tau \, , \, p_\sigma$$

So our phase space is:

$$X = \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2$$
$$\downarrow \qquad \downarrow \qquad \downarrow$$
$$(\tau, \sigma) \qquad (x) \qquad (p_\tau, p_\sigma)$$

In X, there is a submanifold Σ with coordinates

$$\left(\tau, \sigma, x = \phi(\tau, \sigma), p_\tau = \pi_\tau, p_\sigma = \pi_\sigma\right)$$

with

$$\pi_\tau = \frac{\partial \phi}{\partial \tau}, \quad \pi_\sigma = -\frac{\partial \phi}{\partial \sigma}$$

- Define a 2-form on X:

$$\Theta = \frac{1}{2}\left(p_\tau^2 - p_\sigma^2\right) d\tau \wedge d\sigma$$
$$- p_\tau \, dx \wedge d\sigma + p_\sigma \, dx \wedge d\tau$$

Let $\omega = d\Theta$:

$$\omega = \left(p_\tau dp_\tau - p_\sigma dp_\sigma\right) \wedge d\tau \wedge d\sigma$$
$$- dp_\tau \wedge dx \wedge d\sigma + dp_\sigma \wedge dx \wedge d\tau$$

ω is clearly closed

ω is non degenerate:

Let $v = v_\tau \frac{\partial}{\partial \tau} + v_\sigma \frac{\partial}{\partial \sigma} + v_x \frac{\partial}{\partial x}$

$+ v_{p_\tau} \frac{\partial}{\partial p_\tau} + v_{p_\sigma} \frac{\partial}{\partial p_\sigma}$

Assume $\omega(v, -, -) = 0$; check that $v = 0$.

So ω is a symplectic 3-form.

Given a function we can find a bivector field from ω s.t. the bivector field corresponds to solutions of the wave equation.

We want to show that Ham(X) contains observables:

Let $v_\tau = \frac{\partial}{\partial \tau}$ (coordinate vector in the time direction)

Define a 1-form $\mathcal{H} := \Theta(v_\tau, -)$

$$\mathcal{H} = \frac{1}{2}(p_\tau^2 - p_\sigma^2)\, d\sigma - p_\sigma\, dx$$

What is $\mathrm{Lie}_{v_\tau} \Theta$?

$$\mathrm{Lie}_{v_\tau} \Theta = 0$$

Using the Weil identity:

$$\mathrm{Lie}_{v_\tau}\Theta = d\, i_{v_\tau}\Theta + i_{v_\tau} d\Theta = 0$$

but $\quad i_{v_\tau}\theta = \theta(v_\tau, -) =: \mathcal{H} \quad$ & $\quad d\theta = \omega$

which implies

$$d\mathcal{H} + i_{v_\tau}\omega = 0$$

$$\Longrightarrow \quad d\mathcal{H} = -i_{v_\tau}\omega$$

So $\mathcal{H} \in \mathrm{Ham}(X)$

We want to show this is in the set of observables:

- Pick a "slice" of Σ that is at constant time τ_0; called Σ_{τ_0}

  A point on Σ_{τ_0} is $(\tau_0, \sigma, \phi(\tau_0,\sigma), \pi_\tau(\tau_0,\sigma), \pi_\sigma(\tau_0,\sigma))$

  &

$$\mathcal{H} = \frac{1}{2}\left(\pi_\tau^2 - \pi_\sigma^2\right) - \pi_\sigma\, d\phi$$

  & $\quad d\phi = \dfrac{\partial\phi}{\partial\sigma}\, d\sigma + \dfrac{\partial\phi}{\partial\tau}\, d\tau$

  but $d\tau = 0$

  So $\quad d\phi = -\pi_\sigma\, d\sigma.$

Then

$$\mathcal{H} = \left(\frac{1}{2}\left(\Pi_\tau^2 - \Pi_\sigma^2\right) + \Pi_\sigma^2\right) d\sigma$$

$$= \frac{1}{2}\left(\underbrace{\Pi_\tau^2}_{= \left(\frac{\partial\phi}{\partial\tau}\right)^2} + \underbrace{\Pi_\sigma^2}_{= \left(\frac{\partial\phi}{\partial\sigma}\right)^2}\right) d\sigma$$

$$\int_{\Sigma_{\tau_0}} \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial\tau}\right)^2 + \left(\frac{\partial\phi}{\partial\sigma}\right)^2\right] d\sigma = \text{energy of string at time } \tau_0$$

Lastly,

$$\{H, H\} = \mathrm{Lie}_{v_\tau} \mathcal{H}$$

$$= d(\Theta(v_\tau, v_\tau)) + \omega(v_\tau, v_\tau, -)$$

$$= 0$$

So, energy is conserved.