Last time we talked about multisymplectic geometry.

Defn: A multisymplectic 3-form ω is a 3-form on a C^∞-manifold X s.t.

(1) $d\omega = 0$ (closed)

(2) $\forall v \in T_x X$, $\omega(v, -, -) = 0 \iff v = 0$
(non degenerate)

We call (X, ω) a multisymplectic manifold of degree 3.

Defn: A 1-form $\alpha \in \Omega^1(X)$ is Hamiltonian if $\exists\, v \in \mathrm{Vect}(X)$ s.t. $d\alpha = -\omega(v, -, -)$.

This set of 1-forms is denoted $\mathrm{Ham}(X)$.

Let $\{\cdot,\cdot\} : \mathrm{Ham}(X) \times \mathrm{Ham}(X) \longrightarrow \mathrm{Ham}(X)$ be the bracket $\{\alpha, \beta\} = \mathrm{Lie}_{v_\alpha} \beta$.

& let $\{\cdot,\cdot\}' : \mathrm{Ham}(X) \times \mathrm{Ham}(X) \longrightarrow \mathrm{Ham}(X)$ be the bracket $\{\alpha, \beta\}' = \omega(v_\alpha, v_\beta, -)$.

In the 2-symplectic case these brackets coincide & give Poisson algebra structure. Now we have the relation

$$\{\alpha, \beta\} = \{\alpha, \beta\}' + d(i_{v_\alpha}\beta).$$

So they differ by an exact 1-form. Both of these brackets fail to give Ham(X) a Poisson algebra structure.

Poisson algebra	$\{\alpha,\beta\} = \mathrm{Lie}_{v_\alpha}\beta$	$\{\alpha,\beta\}' = \omega(v_\alpha, v_\beta, -)$
closed w.r.t. to bracket	✓	✓
anti-symmetry	No! $\{\alpha,\beta\} = -\{\beta,\alpha\} + d(i_{v_\alpha}\beta + i_{v_\beta}\alpha)$	✓
Jacobi identity	✓	No $\{\alpha,\{\beta,\gamma\}'\}' = \{\{\alpha,\beta\}',\gamma\}' + \{\beta,\{\alpha,\gamma\}'\}' + d(\omega(v_\alpha, v_\beta, v_\gamma))$
derivation property	No way!	No way!

So Ham(X) with $\{\cdot,\cdot\}$ or $\{\cdot,\cdot\}'$ is <u>not</u> an object in the category of Lie algebras. They are actually objects in the category of Lie 2-algebras.

2-vector spaces

Defn: (Baez, Crans (HDA 6) 2004):

A 2-vector space is a category internal in Vect

(i.e. the category of vector spaces over k).

Ehresman defined the concept of an internal category in the 1960's.

Defn: Let K be a category. An internal category X in K consists of:

(1) objects X_0 which comprise an object of K

(2) morphisms X_1 which comprise an object of K.

with

(3) a source function $s: X_1 \longrightarrow X_0$ which is a morphism in K

(4) a target function $t: X_1 \longrightarrow X_0$ which is a morphism in K.

(5) an identity assigning map $id : X_0 \longrightarrow X_1$ which is a morphism in K.

(6) a well defined composition:

$$\circ : X_1 \times_{X_0} X_1 \longrightarrow X_1$$

which is a morphism in K.

s.t. all diagrams commute.

<u>Defn</u>: Given categories X & X' internal in K, an <u>internal functor</u> $F : X \longrightarrow X'$ consists of:

(1) a morphism $F_0 : X_0 \longrightarrow X_0'$ in K

(2) a morphism $F_1 : X_1 \longrightarrow X_1'$ in K

s.t. the relevant diagrams commute.

We have a good definition for internal natural transformation here as well.

<u>Prop</u>: (Baez, Crans)

2 - Vect is a 2-category with

(1) 2-vector spaces as objects

(2) linear functors as morphisms

(3) linear natural transformations as 2-morphisms

We want to construct a 2-vector space from Ham(X):

(1) take as objects : Ham(X)

(this is a vector space)

(2) take as morphisms : $\mathrm{Ham}(X) \oplus C^\infty(X)$

(this is a vector space)

∴ these morphisms are defined as:

for $\alpha \in \mathrm{Ham}(X)$, $f \in C^\infty(X)$

$$\alpha \xmapsto{(\alpha, f)} \alpha + df$$

We need to check that $\alpha \in \mathrm{Ham}(X)$, $f \in C^\infty(X)$ implies that $\alpha + df \in \mathrm{Ham}(X)$:

We know $d\alpha = -\omega(v_\alpha, -, -)$

$$\therefore\ d(\alpha + df) = d\alpha + d^2 f = -\omega(v_\alpha, -, -)$$

The Hamiltonian vector field of $\alpha + df$ is v_α.

We have composition of morphisms as follows:

$$\begin{tikzcd}
\alpha \arrow[r, mapsto, "{(\alpha, f)}"] \arrow[dr, mapsto, "{(\alpha, f+g)}"'] & \alpha + df \arrow[d, mapsto, "{(\alpha + df, g)}"] \\
& \alpha + df + dg
\end{tikzcd}$$

We denote this category $\underline{\mathrm{Ham}(X)}$.

We can represent this structure $\mathrm{Ham}(X)$ as:

$$C^\infty(X) \xrightarrow{d} \mathrm{Ham}(X)$$

a 2-term co-chain complex.

We define $2\mathrm{Term}$ to be the category of 2-term co-chain complexes with chain maps as morphisms & even further it is a 2-category with chain homotopies as 2-morphisms.

Thm: (Baez, Crans)

2 Term & 2 Vect are 2-equivalent.

Given a 2-vector space V,

take $C_1 = V_0 = Ob(V)$

& $C_0 = \ker(s) \subseteq V_1 = Mor(V)$

$d = t|_{C_0}$

which gives us a 2-term (co)-chain complex.

Moving on to Lie 2-algebras ...

If $\underline{Ham(X)}$ is a category, how should we think of the brackets $\{\cdot,\cdot\}$, $\{\cdot,\cdot\}'$ as acting on $\underline{Ham(X)}$?

Answer: They are bilinear functors!

Defn: (Roytenberg, (2007))

A <u>Lie</u> <u>2-algebra</u> is a category L internal in Vect equipped with:

(1) a bilinear functor: (the bracket)

$$\{\cdot,\cdot\} : L \otimes L \longrightarrow L$$

(2) a bilinear natural transformation (the alternator)

$$S : \{\cdot,\cdot\} \Longrightarrow -\{\cdot,\cdot\} * \sigma$$

(3) a trilinear natural transformation (the Jacobiator)

$$J : \{\cdot,\{\cdot,\cdot\}\} \Longrightarrow \{\{\cdot,\cdot\},\cdot\} + \{\cdot,\{\cdot,\cdot\}\} * \sigma_{12}$$

s.t. a whole bunch of diagrams commute.

Remark: The Lie 2-algebra is <u>semistrict</u> if $S = 1$ & <u>hemistrict</u> if $J = 1$. & <u>strict</u> if both semi & hemi.