Recall from last time that

- symplectic structures are defined by closed, non-degenerate 2-forms on X.
- these structures give Poisson algebras on $C^\infty(X)$
- this is a nice language for classical mechanics

Now, we ask:

What is so special about "2"?

Can we define a "symplectic structure" from a p-form?

What kind of algebraic structure would this give to $(p-2)$-forms?

What kind of physical system (if any) could this structure describe?

Some references:

- A. Ibort: "multisymplectic geometry: generic & exceptional" (2000)

- Cantrijn, Ibort, et al.: "On the geometry of multisymplectic manifolds" (1998)

- Baez, Hoffnung, Rogers: (in prep.)

<u>Defn:</u> A p-form ω on a C^∞-manifold X is <u>multisymplectic</u> if:

(1) it is closed: $d\omega = 0$

(2) it is non-degenerate:

$$\text{for } v \in T_x X, \quad \omega(v, -, -) = 0 \iff v = 0$$

What kind of manifolds have such structures?

Some examples:

1) Any orientable manifold X with $\dim X = n$ has a multisymplectic form of degree n. (It is the volume form!)

Say $\mathbb{R}^3$: volume form $dx \wedge dy \wedge dz$

2) Let X be a C^∞-manifold & T^*X be the cotangent bundle. Let $\Lambda^p(T^*X)$ be the p^{th}-exterior power of T^*X.

We get a fiber bundle:

$$\begin{array}{c} \Lambda^p(T^*X) \\ \big\downarrow \pi \\ X \end{array}$$

& there is a canonical p-form Θ on $\Lambda^p(T^*X)$:

- a "point" in $\Lambda^p(T^*X)$ is (x, α), $x \in X$, $\alpha \in \Omega^p(X)$

We define Θ at a point (x, α) by pulling back over π, $\pi^* \alpha$.

Then we get a canonical $(p+1)$-multisymplectic form

$$\omega = d\Theta$$

In local coordinates:

$$\Theta = y_{i_1 \cdots i_p}\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

therefore,

$$\omega = dy_{i_1 \cdots i_p} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

We can see ω is closed & is also nondegenerate since $\{dx^{i_j}\}$ are linearly independent.

3) Let G be a semi-simple Lie group with Lie algebra $\mathfrak{g}$.

- $\mathfrak{g}$ semi-simple $\Longleftrightarrow$ there is an inner product
  $\langle \cdot , \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$
  (the killing form)

- there is an isomorphism between $\mathfrak{g}$ & left-invariant vector fields on G. This gives $\langle \cdot , \cdot \rangle$ meaning on $\mathrm{Vect}(G)$.

$$\Longrightarrow \Omega(\alpha, \beta, \gamma) := \langle \alpha, [\beta, \gamma] \rangle + \langle \beta, [\gamma, \alpha] \rangle + \langle \gamma, [\alpha, \beta] \rangle$$

is a multi-symplectic 3-form on G.

4) A Calabi-Yau manifold X with $\dim_{\mathbb{R}} X = 2n$ admits a multisymplectic n-form.

Classification of multisymplectic manifolds is still open.

Defn: Let $\alpha \in \Omega^{p-2}(X)$, where (X, ω) is a multisymplectic manifold of degree p. We say α is a Hamiltonian $(p-2)$-form if $\exists\, v_\alpha \in \mathrm{Vect}(X)$ s.t. $d\alpha = -i_{v_\alpha}\omega$. We denote this set $\mathrm{Ham}(X, \omega)$; we call v_α a Hamiltonian vector field.

The map

$$\begin{aligned} T_x X &\longrightarrow \Omega^{p-1}(X) \\ v_\alpha &\longmapsto i_{v_\alpha}\omega \end{aligned}$$

is injective, but not surjective in general!

$$\dim T_x X = n \qquad \dim \Omega^{p-1}(X) = \binom{n}{p-1}$$

Note: $\mathrm{Ham}(X, \omega)$ is a vector space.

<u>Prop:</u> For $\alpha \in \mathrm{Ham}(X, \omega)$, ω is invariant along the flow generated by v_α.

Proof: $\mathrm{Lie}_{v_\alpha} \omega = d i_{v_\alpha} \omega + i_{v_\alpha} d\omega = d i_{v_\alpha} \omega = -d^2 \alpha = 0$ ☒

Henceforth, we set $p = 3$. So ω is a 3-form & $\mathrm{Ham}(X)$ is a vector space of Hamiltonian 1-forms.

From "regular" symplectic geometry ($p = 2$): for $f, g \in C^\infty(X)$, we defined $\{f, g\} = \omega(v_f, v_g)$ & we showed that

$\{f, g\} = \mathrm{Lie}_{v_f} g$.

For $p = 3$: we have a good candidate for a bracket on some structure given to the Hamiltonian 1-forms. We would like $\mathrm{Ham}(X)$ to be a Poisson algebra.

<u>Defn:</u> $\{\cdot, \cdot\} : \mathrm{Ham}(X) \times \mathrm{Ham}(X) \longrightarrow \mathrm{Ham}(X)$

$$(\alpha, \beta) \longmapsto \mathrm{Lie}_{v_\alpha} \beta$$

This implies:

$$\{\alpha, \beta\} = i_{v_\alpha} \beta + d(i_{v_\alpha} \beta)$$

Since $d\beta = -i_{v_\beta} \omega$,

$$\{\alpha, \beta\} = -i_{v_\alpha} i_{v_\beta} \omega + d(i_{v_\alpha} \beta)$$

$$= -\omega(v_\beta, v_\alpha, -) + d(i_{v_\alpha} \beta)$$

$$= \omega(v_\alpha, v_\beta, -) + \underbrace{d(i_{v_\alpha} \beta)}_{\text{exact 1-form}}$$

So we could define another bracket

$$\{\cdot, \cdot\}' : \mathrm{Ham}(X) \times \mathrm{Ham}(X) \longrightarrow \mathrm{Ham}(X)$$

$$\alpha, \beta \longmapsto \omega(v_\alpha, v_\beta, -)$$

& $\{\alpha, \beta\}' \neq \{\alpha, \beta\}$. These are equal if $p = 2$.

To get a Poisson algebra on $\mathrm{Ham}(X)$ we need:

(1) closure: $\{\alpha, \beta\} \in \mathrm{Ham}(X)$

(2) antisymmetry: $\{\alpha, \beta\} = -\{\beta, \alpha\}$

(3) Jacobi identity:

$$\{\alpha, \{\beta, \gamma\}\} = \{\{\alpha, \beta\}, \gamma\} + \{\beta, \{\alpha, \gamma\}\}$$

(4) Derivation property:

$$\{\alpha, \beta\gamma\} = \{\alpha, \beta\}\gamma + \beta\{\alpha, \gamma\}$$

Since we do not know how to multiply 1-forms, we know that (4) cannot hold.

Proof of closure:

$\alpha, \beta \in \mathrm{Ham}(X)$: show $\exists\, v \in \mathrm{Vect}(X)$ s.t.

$$d\{\alpha, \beta\} = -i_v \omega$$

By definition, $\{\alpha, \beta\} = \mathrm{Lie}_{v_\alpha} \beta$. So,

$$\begin{aligned}
d\{\alpha, \beta\} &= d\,\mathrm{Lie}_{v_\alpha} \beta \\
&= \mathrm{Lie}_{v_\alpha} d\beta \\
&= \mathrm{Lie}_{v_\alpha} i_{v_\beta} \omega \\
&= -\left(i_{[v_\alpha, v_\beta]} \omega + i_{v_\beta} \mathrm{Lie}_{v_\alpha} \omega\right) \\
&= -i_{[v_\alpha, v_\beta]} \omega
\end{aligned}$$

$\boxtimes$