We recall some identities & definitions from last time:

If v, w are vector fields & α is a p-form:

(1) $\mathrm{Lie}_v \alpha = i_v d\alpha + d i_v \alpha$

(i_v is the interior product & d is the exterior derivative)

(2) $\mathrm{Lie}_v d\alpha = d\,\mathrm{Lie}_v \alpha$

(3) $\mathrm{Lie}_v i_w \alpha = i_{[v,w]} \alpha + i_w \mathrm{Lie}_v \alpha$

Also from last time:

- We defined a symplectic manifold:

  X is a smooth (C^∞-) manifold & ω is a 2-form, with $d\omega = 0$ (closed) & if $v \in T_x(X)$ & $\forall u \in T_x(X)$, $\omega(v,u) = 0 \Rightarrow v = 0$ (non-degeneracy)

- ω gives an isomorphism $\forall x \in X$

$$\omega : T_x(X) \longrightarrow T_x^*(X)$$
$$v \longmapsto \omega(v, -)$$

- Given a function $f \in C^\infty(X)$, we defined $v_f \in \mathrm{Vect}(X)$ to be:

$$df = -i_{v_f}\omega \quad \text{(note the sign change)}$$

This v_f is a Hamiltonian vector field.

- Symplectic structure makes $C^\infty(X)$ a Poisson algebra (Lie algebra + extra property)

$$\{f, gh\} = \{f,g\}h + g\{f,h\}$$

What is $\{\cdot,\cdot\}$?

$$f, g \in C^\infty(X) \quad , \quad \{f,g\} = \omega(v_f, v_g)$$

From this definition we get some identities:

- $\{f,g\} = \omega(v_f, v_g)$

$$df = -i_{v_f}\omega = -\omega(v_f, -)$$

$$dg = -i_{v_g}\omega = -\omega(v_g, -)$$

$$dg(v_f) = -\omega(v_g, v_f) = \omega(v_f, v_g)$$

- $\{f,g\} = dg(v_f) = v_f(g)$

- $\{f,g\} = i_{v_f} dg$

Note: $\mathrm{Lie}_{v_f} g = i_{v_f} dg + d(i_{v_f} g)$; $i_{v_f} g = 0$

- $\{f,g\} = \mathrm{Lie}_{v_f} g$

Lie algebras

(1) Lie algebra: $\mathrm{Vect}(X)$

$[\cdot,\cdot]$ = Lie bracket of vector fields

(2) Poisson algebra: $(C^\infty(X), \{\cdot,\cdot\})$

<u>Proposition</u>: The set of all Hamiltonian vector fields $\mathrm{Vect}_H(X)$ is a Lie subalgebra of $\mathrm{Vect}(X)$. Furthermore,

$$\begin{aligned} C^\infty(X) &\xrightarrow{\phi} \mathrm{Vect}_H(X) \\ f &\longmapsto v_f \end{aligned}$$

is a Lie algebra homomorphism.

Proof: Show $v_f, v_g \in \mathrm{Vect}_H(X) \Longrightarrow [v_f, v_g] \in \mathrm{Vect}_H(X)$ by showing $\exists\, F \in C^\infty(X)$ s.t.

$$dF = -i_{[v_f, v_g]}\,\omega$$

From our identities & Liouville's theorem from last time, we have

$$\mathrm{Lie}_{v_f}\, i_{v_g}\,\omega = i_{[v_f,v_g]}\,\omega + i_{v_g}\,\mathrm{Lie}_{v_f}\,\omega = i_{[v_f,v_g]}\,\omega$$

Note: $i_{v_g}\omega = -dg$

$$\mathrm{Lie}_{v_f}(-dg) = i_{[v_f, v_g]}\omega$$

$$-d\,\mathrm{Lie}_{v_f} g = i_{[v_f, v_g]}\omega$$

$$d\{f,g\} = -i_{[v_f, v_g]}\omega$$

So this implies

$$\{f,g\} \longmapsto [v_f, v_g]$$

which implies

$$\phi(\{f,g\}) = [\phi(f), \phi(g)]. \qquad \square$$

Question: What is the kernel of ϕ?

Answer: $f \in \ker\phi \implies v_f = 0 \implies -i_{v_f}\omega = 0$
$= \omega(0, -)$

$\implies df = 0$

$\implies f$ is locally constant.

So, $\ker\phi = \mathbb{R}^{n_c}$, n_c = # of connected components of X.

Example: $\mathbb{R}^2$, $\omega = dp \wedge dq$

$f \in C^\infty(\mathbb{R}^2)$, what is v_f?

$$v_f = v_f^q \frac{\partial}{\partial q} + v_f^p \frac{\partial}{\partial p}$$

$$df = \frac{\partial f}{\partial q} dq + \frac{\partial f}{\partial p} dp$$

$$df = -\omega(v_f, -)$$

$$= \left(-v_f^q \frac{\partial}{\partial q} dp \wedge dq\right) - \left(v_f^p \frac{\partial}{\partial p} dp \wedge dq\right)$$

$$= v_f^q \frac{\partial}{\partial q} dq \wedge dp - v_f^p dq$$

$$= v_f^q dp - v_f^p dq$$

equating coefficients, we have

$$\frac{\partial f}{\partial q} = -v_f^p, \quad \frac{\partial f}{\partial p} = v_f^q$$

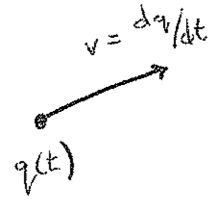
in $\mathbb{R}^2$,

$$v_f = \frac{\partial f}{\partial p}\frac{\partial}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial}{\partial p}$$

Then,

$$\{f, g\} = v_f(g) = \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p}$$

Hamiltonian mechanics:



$$F = ma \quad , \quad a = \frac{d^2 q}{dt^2}$$

$$F = m \frac{d^2 q}{dt^2} \quad , \quad 2^{nd} \text{ order ODE}$$

Let the force be minus the gradient of a potential function

$$V : \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$- \nabla V = m \frac{d^2 q}{dt^2}$$

- define momentum $p = mv = m \frac{dq}{dt}$
- 2 first order equations:

$$\frac{dq}{dt} = \frac{p}{m} \quad , \quad \frac{dp}{dt} = - \nabla V$$

define $H : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$

$$H(q, p) = \underset{\substack{\uparrow \\ \text{kinetic}}}{\frac{p^2}{2m}} + \underset{\substack{\uparrow \\ \text{potential}}}{V(q)}$$

$$\frac{\partial H}{\partial q} = \nabla V \, , \quad \frac{\partial H}{\partial p} = \frac{p}{m}$$

$$\Longrightarrow \quad \frac{dq}{dt} = \frac{\partial H}{\partial p} \, , \quad \frac{dp}{dt} = - \frac{\partial H}{\partial q}$$

These are Hamilton's equations.

(a solution is : $q(t)$, $p(t)$)

Constructing a symplectic structure:

Let $\mathbb{R}$ be our configuration space. This implies that $\mathbb{R}^2$ is the space of position & momentum, $(q,p) \in \mathbb{R}^2$ (phase space).

The symplectic form is $\omega = dp \wedge dq$ & the Hamiltonian $H \in C^\infty(\mathbb{R}^2)$.

What is v_H?

$$v_H = \frac{\partial H}{\partial p} \frac{\partial}{\partial q} - \frac{\partial H}{\partial q} \frac{\partial}{\partial p}$$

What is the flow of v_H?

Find a curve $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^2$

$$t \longmapsto (q(t), p(t))$$

Then,

$$\frac{d\gamma}{dt} = \frac{dq}{dt}\frac{\partial}{\partial q} + \frac{dp}{dt}\frac{\partial}{\partial p}$$

if v_H is the vector field tangent to γ, then

$$\frac{dq}{dt} = \frac{\partial H}{\partial t} \quad , \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

Physical meaning of $\{\cdot,\cdot\}$:

$f \in C^\infty(\mathbb{R}^2)$, what is $\{H, f\}$?

$$\{H, f\} = \mathrm{Lie}_{v_H} f . = \frac{df}{dt}(q(t), p(t))$$

Conserved quantities:

"conservation of energy"

$\dfrac{dH(q(t),p(t))}{dt} = 0$, which we have since $\{H, H\} = 0$.