Multisymplectic geometry & Lie n-algebras

Symplectic vector spaces:

Defn: A fin-dim symplectic vector space is a vector space V & a bilinear form ω that is anti-symmetric – i.e. $\omega(v,v') = -\omega(v',v)$ & non-degenerate – i.e. if $\omega(v,v') = 0$, $\forall v' \in V \Longrightarrow v = 0$.

Example: Let Q be a real vector space with $\dim(Q) = n$. Then $V = Q \oplus Q^*$ is symplectic with $\omega((q,p),(q',p')) = \frac{1}{2}(p(q') - p'(q))$.

Prop: All fin-dim symplectic vector spaces "look" like this.

Note: $\dim(V) = 2n$

<u>Defn</u>: Let $W \subseteq V$. The <u>symplectic complement</u>

$$W^{\perp} = \{ v \in V \mid \omega(u, v) = 0, \ \forall u \in W \}$$

<u>Lemma</u>: $\dim(V) = \dim(W) + \dim(W^{\perp})$

<u>Defn</u>: Let $W \subseteq V$; we say W is

(1) isotropic if $W \subseteq W^{\perp}$

(2) coisotropic if $W^{\perp} \subseteq W$

(3) Lagrangian if $W = W^{\perp}$

<u>Prop</u>: Every fin-dim symplectic vector space contains a Lagrangian subspace.

<u>Cor</u>: Every fin-dim symplectic vector space is even dimensional.

Symplectic Manifolds:

Defn: Let X be a real C^∞-manifold. A symplectic structure on X is a closed non-degenerate 2-form $\omega \in \Omega^2(X)$.

Notes:
- closed means $d\omega = 0$.
- $\omega \in \Omega^2(X)$, then $\forall\, x \in X$ we get ω_x on $T_x(X)$; $(T_x(X), \omega_x)$ is a symplectic vector space.
- $\dim(X)$ is even

Non-degeneracy implies that

$$\omega^{\wedge n} = \omega \wedge \cdots \wedge \omega$$

is a non-zero $2n$-form, which implies X is orientable.

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Example: $X = \mathbb{R}^{2n}$ coords $\{q^i\}_{i=1}^{2n}$

$$\omega = \sum_{i=1}^{n} dq^i \wedge dq^{n+i}$$

Example: Let X be a C^∞ - manifold with $\dim(X) = n$.

The cotangent bundle T^*X is symplectic.

$\{q^i\}$ are coordinates on X & $\{p_i\}$ be the fiber coordinates: we can define the canonical 1-form Θ by

$$\Theta := \sum_{i=1}^{n} p_i dq^i$$

then

$$d\Theta = \sum_{i=1}^{n} dp_i \wedge dq^i = \omega.$$

Thm (Darboux): There exists a neighborhood U of any point x in a symplectic manifold X & coordinate system $\{q^i, p_i\}$ such that

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq^i.$$

Global fact:

- The fact that $d\omega = 0$ implies

$$[\omega] \in H^2_{DR}(X, \mathbb{R})$$

Example: If X is closed (compact & no boundary) then $H^2_{DR}(X, \mathbb{R})$ is non-trivial.

The isomorphism generated by ω_x, $\forall x \in X$:

$$\begin{aligned} T_x X &\xrightarrow{\sim} T_x^* X \\ v &\longmapsto \omega_x(v, -) \end{aligned}$$

Useful identities:

- Defn: (interior product)

  Let $v \in \mathrm{Vect}(X)$

$$\begin{aligned} i_v : \Omega^p(X) &\longrightarrow \Omega^{p-1}(X) \\ \alpha &\longmapsto \alpha(v, -, -, \ldots, -) \end{aligned}$$

(From here on we assume X compact.)

Let $\mathrm{Lie}_v\,\alpha$ be the Lie derivative of a p-form α along the flow generated by the vector field v.

- $\mathrm{Lie}_v\,\alpha = i_v d\alpha + d i_v \alpha$

- $\mathrm{Lie}_v\, d\alpha = d\,\mathrm{Lie}_v\,\alpha$

  Proof: $$\begin{aligned}\mathrm{Lie}_v\, d\alpha &= i_v d d\alpha + d i_v d\alpha \\ &= d i_v d\alpha \\ &= d i_v d\alpha + d d i_v \alpha \\ &= d\,\mathrm{Lie}_v\,\alpha\end{aligned}$$

- $\mathrm{Lie}_v\, i_w\, \alpha = i_{[v,w]}\,\alpha + i_w\,\mathrm{Lie}_v\,\alpha$,

  where $[\cdot,\cdot]$ is Lie bracket on $\mathrm{Vect}(X)$.

- $\mathrm{Lie}_{[v,w]}\,\alpha = \mathrm{Lie}_v\,\mathrm{Lie}_w\,\alpha - \mathrm{Lie}_w\,\mathrm{Lie}_v\,\alpha$

Hamiltonian vector fields:

Let $f \in C^\infty(X)$. $df \in \Omega^1(X)$ (is a 1-form) implies $\exists\ v_f \in \mathrm{Vect}(X)$ s.t. $\omega(v_f, -) = df$,

i.e.,

$$df = i_{v_f}\,\omega$$

<u>Defn:</u> Let $v \in \mathrm{Vect}(X)$. v is <u>Hamiltonian</u> if

$$i_v \omega = df \quad \text{for some } f \in C^\infty(X).$$

<u>Prop:</u> Flows of Hamiltonian vector fields preserve the symplectic form, i.e.

$$\mathrm{Lie}_{v_f} \omega = 0.$$

Proof:
$$\begin{aligned} \mathrm{Lie}_{v_f} \omega &= d\, i_{v_f} \omega + i_{v_f} d\omega \\ &= d\,df + i_{v_f}(0) \\ &= 0 \end{aligned}$$

We can ask if the converse is true.

$$\mathrm{Lie}_v \omega = 0 \implies v \text{ is Hamiltonian?}$$

$$\mathrm{Lie}_v \omega = d(i_v \omega) = 0$$

So, if $H^1_{DR}(X, \mathbb{R}) = 0$.

Poisson algebras:

Defn: A Poisson algebra is a commutative algebra $\mathbb{A}$ equipped with a bilinear operation

$$\{\cdot,\cdot\}: \mathbb{A}\times\mathbb{A} \longrightarrow \mathbb{A}$$

with the following properties:

(1) antisymmetry: $\{a,b\} = -\{b,a\}$

(2) Jacobi identity: $\{a,\{b,c\}\} = \{\{a,b\},c\} + \{b,\{a,c\}\}$

(3) derivation: $\{a,bc\} = \{a,b\}c + b\{a,c\}$

Prop: Let (X,ω) be a symplectic manifold. $C^\infty(X)$ is a Poisson algebra with the following bracket:

$$\{f,g\} = \omega(v_f, v_g)$$

where v_f, v_g are Hamiltonian.

We have $\{f,g\} = df(v_g) = v_g(f) = -\mathrm{Lie}_{v_f} g$