Lie 2-algebras from 2-plectic geometry

Chris Rogers
joint with John Baez and Alex Hoffnung

Department of Mathematics
University of California, Riverside

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Motivation from Physics

Classical Particles

The theory of classical point particles is a theory of **0-dimensional objects** which can be described by:

- paths in a smooth manifold X,
- a closed non-degenerate **2-form** ω on X called the **symplectic structure** and,
- a set of **smooth functions** on X called **observables**.

The symplectic structure ω makes the set of observables into a **Poisson algebra**.

Motivation from Physics

Classical Strings

The classical theory of strings is a theory of **1-dimensional objects.** Recent work suggests that it can be described by:

- surfaces, or **world-sheets** in a smooth (finite dimensional) manifold X,
- a closed, non-degenerate **3-form** ω called the **2-plectic structure** and,
- a set of **1-forms** on X called **observables**.

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Question: What kind of algebraic structure does the 2-plectic structure ω give the set of observables?

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Question: What kind of algebraic structure does the 2-plectic structure ω give the set of observables?

- ω makes the observables into a **Lie 2-algebra**.
- This is a kind of **categorification** that naturally reflects the increase in dimension.

Symplectic Geometry

Definition of a Symplectic Structure

A **symplectic structure** on a smooth manifold X is a smooth **2-form** ω that is closed and non-degenerate:

$$d\omega = 0,$$

$$\forall v \in T_x X \quad \omega(v, \cdot) = 0 \Rightarrow v = 0.$$

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The non-degeneracy of ω implies the **isomorphism**:

$$\begin{array}{ccc} T_x X & \xrightarrow{\sim} & T_x^* X \\ v & \mapsto & \omega(v, \cdot). \end{array}$$

Hence given $f \in C^\infty(X)$, there exists a unique vector field v_f such that

$$df = -\omega(v_f, \cdot).$$

Definition

If (X, ω) is a symplectic manifold, we say $v \in \mathrm{Vect}(X)$ is **Hamiltonian** iff there exists an $f \in C^\infty(X)$ such that

$$df = -\omega(v, \cdot).$$

ω gives us the well-defined bilinear map:

$$\{\cdot, \cdot\} : C^\infty(X) \times C^\infty(X) \to C^\infty(X),$$

where

$$\{f, g\} = \omega(v_f, v_g)$$

ω makes $C^{\infty}(X)$ into a **Poisson algebra** $\left(C^{\infty}(X), \{\cdot,\cdot\}\right)$:

- Antisymmetry
$$\{f,g\} = -\{g,f\},$$
- Jacobi Identity
$$\{f,\{g,h\}\} = \{\{f,g\},h\} + \{g,\{f,h\}\},$$
- Leibniz's Law
$$\{f,gh\} = \{f,g\}h + g\{f,h\}.$$

A **2-plectic structure** on a smooth manifold X is a smooth **3-form** ω that is **closed and non-degenerate**:

$$d\omega = 0,$$

$$\forall v \in T_x X \quad \omega(v, \cdot, \cdot) = 0 \Rightarrow v = 0.$$

ω is also referred to as a **multisymplectic 3-form**.

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The theory of **multisymplectic geometry** originated with Weyl's work on the calculus of variations, and is still undergoing much development.

For example: Cantrijn, Ibort, and DeLeón (1998), Gotay, Isenberg, Marsden, and Montgomery (1998), and Hélein (2001) and Forger, Paufler, and Römer (2004).

Example 1

Let M be a smooth manifold. Let X be the bundle $\Lambda^2 T^*M \xrightarrow{\pi} M$.

Then X has a **canonical 2-form**:

$$\theta(v_1, v_2) = x(d\pi(v_1), d\pi(v_2)),$$

where v_1, v_2 are tangent vectors at the point $x \in X$.

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$\omega = d\theta$ is a 2-plectic structure on X

X is related to the **phase space for classical strings**.

Example 2

Any **compact simple Lie group** G is a 2-plectic manifold with 2-plectic form:

$$\nu_k(v_1, v_2, v_3) = k\langle v_1, [v_2, v_3]\rangle$$

where v_i are tangent vectors in $\mathfrak{g}$, $\langle \cdot, \cdot \rangle$ is the Killing form, and k is non-zero.

- ν_k is invariant under left and right translations and therefore closed.
- ν_k is non-degenerate since $\mathfrak{g}$ is simple.

Let (X, ω) be a 2-plectic manifold. From the non-degeneracy of ω we have an **injective** map

$$\begin{aligned} T_x X &\to \Lambda^2 T_x^* X \\ v &\mapsto \omega(v, \cdot, \cdot). \end{aligned}$$

(**Not an isomorphism** in general.)

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Let (X, ω) be a 2-plectic manifold. A 1-form α on X is **Hamiltonian** if there exists a vector field v_α on X such that

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We denote the vector space of Hamiltonian 1-forms as $\mathrm{Ham}(X)$.

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We denote the vector space of Hamiltonian 1-forms as $\mathrm{Ham}\,(X)$.

We say v_α is the **Hamiltonian vector field** corresponding to α.

We can define **a bracket of Hamiltonian 1-forms** similar to the Poisson bracket of functions in the symplectic case:

Definition

Given $\alpha, \beta \in \mathrm{Ham}\,(X)$, the **bracket** $\{\alpha, \beta\}$ is the 1-form given by

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$\mathrm{Ham}\,(X)$ is **closed under the bracket**, but...

$(\mathrm{Ham}\,(X)\,, \{\cdot, \cdot\})$ **is not a Lie algebra.**

2-plectic Geometry

The bracket on $\mathrm{Ham}\,(X)$

The bracket $\{\cdot,\cdot\}$ is **antisymmetric**:

$$\{\alpha,\beta\} = \omega(v_\alpha, v_\beta, \cdot) = -\omega(v_\beta, v_\alpha, \cdot) = -\{\beta,\alpha\},$$

2-plectic Geometry

The bracket $\{\cdot,\cdot\}$ is **antisymmetric**:

$$\{\alpha,\beta\} = \omega(v_\alpha, v_\beta, \cdot) = -\omega(v_\beta, v_\alpha, \cdot) = -\{\beta,\alpha\},$$

but **does not satisfy the Jacobi identity**:

$$\{\alpha,\{\beta,\gamma\}\} + dJ_{\alpha,\beta,\gamma} = \{\{\alpha,\beta\},\gamma\} + \{\beta,\{\alpha,\gamma\}\},$$

where $J_{\alpha,\beta,\gamma} = \omega(v_\alpha, v_\beta, v_\gamma)$.

The identity holds only “up to” an **exact 1-form**.

Definition

A **Lie 2-algebra** is a 2-term chain complex of vector spaces $L = (L_0 \stackrel{d}{\leftarrow} L_1)$ equipped with the following structure:

- a antisymmetric chain map $[\cdot, \cdot]\colon L \otimes L \to L$ called the **bracket**,
- an antisymmetric chain homotopy $J\colon L \otimes L \otimes L \to L$ from the chain map

$$x \otimes y \otimes z \longmapsto [x, [y, z]],$$

to the chain map

$$x \otimes y \otimes z \longmapsto [[x, y], z] + [y, [x, z]]$$

called the **Jacobiator**.

Lie 2-algebras

Definition of a Lie 2-algebra

In addition, the Jacobiator is required to satisfy:

$$\begin{aligned}[x, J(y,z,w)] + J(x,[y,z],w) + J(x,z,[y,w]) + [J(x,y,z),w] \\ +[z, J(x,y,w)] = J(x,y,[z,w]) + J([x,y],z,w) \\ +[y, J(x,z,w)] + J(y,[x,z],w) + J(y,z,[x,w]).\end{aligned}$$

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Dmitry Roytenberg has extended the definition of a Lie 2-algebra to include Lie 2-algebras whose brackets **only satisfy antisymmetry up to isomorphism**.

Given a 2-plectic manifold (X, ω), we can construct a Lie 2-algebra with the underlying 2-term complex:

$$L \quad = \quad \mathrm{Ham}\,(X) \xleftarrow{d} C^{\infty}\,(X)$$

$\mathrm{Ham}\,(X)$ is the space of degree 0 chains, $C^{\infty}\,(X)$ is the space of degree 1 chains, and d is the exterior derivative of functions.

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Note that **any exact form is Hamiltonian**, with 0 as its Hamiltonian vector field.

The bracket $\{\cdot, \cdot\}$ can be extended from $\mathrm{Ham}\,(X) \otimes \mathrm{Ham}\,(X)$ to $L \otimes L$ by setting it to the zero map in all degrees other than 0.

Theorem

If (X, ω) *is a 2-plectic manifold, there is a Lie 2-algebra* $L(X, \omega)$ *where:*

- *the space of 0-chains is* $\mathrm{Ham}(X)$*,*
- *the space of 1-chains is* $C^{\infty}(X)$*,*
- *the differential is the exterior derivative* $d\colon C^{\infty}(X) \to \mathrm{Ham}(X)$*,*
- *the bracket is* $\{\cdot, \cdot\}$*,*
- *the Jacobiator is the linear map* $J\colon \mathrm{Ham}(X) \otimes \mathrm{Ham}(X) \otimes \mathrm{Ham}(X) \to C^{\infty}(X)$ *defined by* $J_{\alpha,\beta,\gamma} = \omega(v_{\alpha}, v_{\beta}, v_{\gamma})$*.*

Lie 2-algebras from 2-plectic Structures

Some Remarks

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Any 2-plectic manifold also gives rise to a Lie-2 algebra $L'(X, \omega)$ whose bracket satisfies the Jacobi identity but satisfies antisymmetry **up to an exact 1-form**.

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Any 2-plectic manifold also gives rise to a Lie-2 algebra $L'(X,\omega)$ whose bracket satisfies the Jacobi identity but satisfies antisymmetry **up to an exact 1-form**.

$L'(X,\omega)$ has the same underlying chain complex as $L(X,\omega)$: $\mathrm{Ham}\,(X) \xleftarrow{d} C^\infty\,(X)$. Its bracket is defined by:

$$\{\alpha,\beta\}' = \mathcal{L}_{v_\alpha}\beta.$$

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$$\{\alpha,\beta\}' = \mathcal{L}_{v_\alpha}\beta.$$

$$\{\alpha,\beta\}' = -\{\beta,\alpha\}' + d\left(\alpha(v_\beta) + \beta(v_\alpha)\right).$$

$\{\alpha, \beta\}$ and $\{\alpha, \beta\}'$ are related by an **exact 1-form**:

$$\{\alpha, \beta\}' = \{\alpha, \beta\} + d\beta(v_\alpha).$$

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The Lie 2-algebra $L'(X, \omega)$ is **isomorphic** to $L(X, \omega)$.

Note that the brackets $\{\cdot, \cdot\}$ and $\{\cdot, \cdot\}'$ are **equal** in the case of **symplectic geometry**.

When the 2-plectic manifold X is the phase space of the classical string, $L(X, \omega)$ contains a 1-form H that is the **Hamiltonian density** of the string.

H gives the **energy** of an infinitesimal piece of string at a fixed time t, and the flow of its Hamiltonian vector field gives the **time evolution**.

Compact Simple Lie Groups

The 2-plectic Structure

Now we consider Lie 2-algebras on **compact simple Lie groups.**

Let G be a compact simple Lie group with Lie algebra $\mathfrak{g}$. Let $\langle \cdot, \cdot \rangle$ be the Killing form on $\mathfrak{g}$ and $k \neq 0$.

Then (G, ν_k) is a 2-plectic manifold with 2-plectic form

$$\nu_k(v_1, v_2, v_3) = k\langle v_1, [v_2, v_3]\rangle$$

where v_i are tangent vectors in $\mathfrak{g}$.

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where v_i are tangent vectors in $\mathfrak{g}$.

Let $\mathfrak{g}^*$ be the set of **left invariant 1-forms** on G.

Let $\mathrm{Ham}\,(G)^L$ be the set of **left invariant Hamiltonian 1-forms**.

Theorem

Every left invariant 1-form on (G, ν_k) is Hamiltonian. That is, $\mathrm{Ham}\,(G)^L = \mathfrak{g}^*$*.*

If α is a left-invariant Hamiltonian 1-form, then its Hamiltonian vector field v_α is an element of the Lie algebra $\mathfrak{g}$ and:

$$\alpha = k\langle v_\alpha, \cdot \rangle.$$

Since the left-invariant smooth functions are constants, we have a 2-term chain complex:

$$L_G = \mathfrak{g}^* \xleftarrow{d=0} \mathbb{R}.$$

Compact Simple Lie Groups

Lie 2-algebras

The bracket $\{\alpha, \beta\} = k\langle v_\alpha, [v_\beta, \cdot]\rangle$ of any two left invariant Hamiltonian 1-forms **is left invariant**.

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Theorem

If G is a compact simple Lie group with Lie algebra $\mathfrak{g}$ and $k \neq 0$, there is a Lie 2-algebra $L(G, k)$ where:

- *the space of 0-chains is $\mathfrak{g}^*$,*
- *the space of 1-chains is $\mathbb{R}$,*
- *the differential is the zero map $d = 0$,*
- *the bracket is $\{\cdot, \cdot\}$,*
- *the Jacobiator is the linear map $J\colon \mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathbb{R}$ defined by $J_{\alpha,\beta,\gamma} = k\langle v_\alpha, [v_\beta, v_\gamma]\rangle$.*

Compact Simple Lie Groups

The string Lie 2-algebra

Given a simple Lie algebra $\mathfrak{g}$ and $k \in \mathbb{R}$ we can construct a Lie 2-algebra $\mathfrak{g}_k$ called the **string Lie 2-algebra** where

- the space of 0-chains is $\mathfrak{g}$,
- the differential is the zero map $d = 0$,
- the bracket is the Lie bracket $[\cdot, \cdot]$ in degree 0 and trivial otherwise,
- the Jacobiator is the 3-cocycle $j(x, y, z) = k\langle x, [y, z]\rangle \in H^3(\mathfrak{g}, \mathbb{R})$.

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Theorem

If G is a compact simple Lie group with Lie algebra $\mathfrak{g}$ and $k \neq 0$, the Lie 2-algebra $L(G, k)$ is isomorphic to $\mathfrak{g}_k$.

Hence we may view $L(G, k)$ as the **purely geometric construction** of the string Lie 2-algebra $\mathfrak{g}_k$.