A TOUR OF GENERAL TOPOLOGY
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1. THE LAUNDRY LIST

1.1. **Metric and topological spaces, open and closed sets.**

1. metric space: open balls $N_\epsilon(x)$, various metrics e.g. discrete metric, $\|\cdot\|_p$ on $\mathbb{R}^n$, sup norm on $\mathrm{Bnd}(X) = \{f \colon X \to \mathbb{R} \mid \forall x \in X |f(x)| < \infty\}$.
2. topological space: discrete topology, in-discrete topology, finitary (or cofinite) topology, $\mathbb{R}_\ell$ (lower limit topology on $\mathbb{R}$)
3. sub-bases and bases
4. comparable topologies: courser (smaller) versus finer (larger)
5. basic Hausdorff and T1 properties
6. closed set, limit point, closure, interior, boundary
7. horrible Cantor set problems e.g. Munkres ex. §27.6
8. subspace topology

1.2. **Maps and homeomorphisms.**

1. Metric spaces: continuity (at least 2 ways), uniform continuity
2. topological spaces: continuity (at least 2 ways)
3. homeomorphism, open, closed, and proper maps
4. isometries between metric spaces

1.3. **Products and quotients.**

1.3.1. *Products.*

1. product topology (finite vs. infinite products), universal property
2. basis for finite products, basis for infinite products
3. closure of a product of subspaces
4. mapping into products
5. the product $\prod_{i=1}^N X_i$ of metric spaces $\{(X_i, d_i)\}_{i=1}^N$ equipped with $d^\infty(x,y) = \max_i d_i(x_i, y_i)$, $d^2(x,y) = \left(\sum_i d_i^2(x_i,y_i)\right)^{1/2}$, and $d^1(x,y) = \sum_i d_i(x_i,y_i)$.

1.3.2. *Quotients.*

1. def. of quotient topology
2. characterization of quotient topology as the strongest topology s.t. the canonical map $\pi \colon X \to X/\sim$ is continuous.
3. def. and various properties of quotient maps
4. Hausdorff and non-Hausdorff quotients

1.4. **Connectedness.**

1. Equivalent definitions of connectedness
2. defs. of path connected, locally connected, and locally path connected
3. **Theorem**: A subset $A \subseteq \mathbb{R}$ with at least 2 elements is connected iff for each $a, b \in A$ with $a < b$, $[a,b] \subseteq A$.
4. characterizing connected subsets of $\mathbb{R}$
5. connected components vs. path-connected components
6. def. of totally disconnected space

1.5. Compactness.

1. cover, subcover, locally finite cover
2. Heine-Borel Theorem
3. compact, sequentially compact, Bolzano-Weierstrass property
4. compact implies limit pt. compact (i.e. Bolzano-Weierstrass)
5. compact and first-countable implies sequentially compact
6. Borel-Lebesgue Theorem: If X is a metric space, then the following are equivalent: X is compact, X is sequentially compact, X has the Bolzano-Weierstrass property.
7. totally bounded metric space
8. finite intersection property (f.i.p)
9. tube lemma, products of compact spaces

1.6. Countability and separation properties.

1.6.1. *Countability.*

1. def. of first-countable, separable, second-countable, Lindelöf
2. second countable $\Rightarrow$ separable
3. second countable $\Rightarrow$ Lindelöf
4. for metric spaces: second countable $\Leftrightarrow$ separable $\Leftrightarrow$ Lindelöf are equivalent

1.6.2. *Separation.*

1. T_1, T_2 (i.e. Hausdorff), T_3 (i.e. T_1 plus regularity), $T_{3\frac{1}{2}}$ (i.e. T_1 plus complete regularity), T_4 (i.e. T_1 plus normal)
2. **Urysohn's Lemma:** If X is T_1, then it is T_4 if and only if for any disjoint closed subspaces A and B, there exists a continuous function $f\colon X \to \mathbb{R}$ s.t. $f|_A = 0$ and $f|_B = 1$.
3. compact and T_2 implies T_4
4. T_4 property is not preserved, in general, with respect to subspaces and products.
5. $T_{i<4}$ properties are preserved with respect to subspaces and products.
6. **Tietze Extension Theorem:** Let X be T_4 and $A \subseteq X$ a closed subspace. If $f\colon A \to \mathbb{R}$ is a continuous function, then there exists a continuous function $\hat{f}\colon X \to \mathbb{R}$ such that $\hat{f}|_A = f$.
7. Tietze extension thm. is logically equivalent to Urysohn's Lemma. Also above we can replace $\mathbb{R}$ everywhere with the closed interval $[a, b]$.
8. Urysohn metrization: If X is second-countable, then X is homeomorphic to a metric space if and only if X is T_3.

1.7. Local compactness/1-pt. compactification.

1.7.1. *Local compactness.*

1. def. of weakly locally compact, and strongly locally compact
2. If X is Hausdorff, then X weakly locally compact iff X strongly locally compact
3. locally compact Hausdorff not necessarily T_4
4. **Theorem:** If X is locally compact and Hausdorff and $Y \subseteq X$ is either open or closed, then Y is locally compact.

1.7.2. *1-pt. compactification.*

1. def. of compactifications
2. restrict ourselves to compactifications of locally compact Hausdorff spaces because: if $f\colon X \rightarrowtail Y$ is a Hausdorff compactification, then $f(X)$ must be Hausdorff and locally compact
3. construction of $1-$pt. compactification $f\colon X \hookrightarrow X \cup \{\infty\}$ and proof that it is a compactification and is a compact Hausdorff space

1.8. **Complete metric spaces and Baire's Theorem.**

(1) Cauchy sequences
(2) complete vs. compact, products and subspaces of complete metric spaces
(3) closed subspaces of complete spaces
(4) nowhere dense sets
(5) first category, second category, meager, non-meager, residual, and co-meager sets
(6) def. of Baire space
(∗) **Theorem**: X is a Baire space if and only if given any countable collection $\{O_n\}$ of dense open subsets in X, the intersection $\cap_n O_n$ is dense in X.

1.9. **Completions of metric spaces.**

(1) definition of a completion of a metric space
(2) definition of Banach space
(3) definition of isometry
(4) **Theorem**: If (X, d) is a metric space, there is an isometry from X into a Banach space.
(5) existence and uniqueness of completions

2. A LIST OF THEOREMS ONE SHOULD KNOW AND BE ABLE TO PROVE

Note: $\star$ denotes a theorem whose proof is a solution to a problem given on a previous qualifying exam.

2.1. **Metric and topological spaces.**

Proposition 2.1. *A metric space is T1 and Hausdorff. Any Hausdorff space is T1.*

Proposition 2.2. *If (X, d) is a metric space, then the collection of open neighborhoods $\mathcal{B} = \{N_\epsilon(x)\}_{x\in X \epsilon>0}$ is a basis for the metric topology.*

Proposition 2.3. *Let X be a space. Let $A \subseteq Y \subseteq X$ be subspaces. Then $\bar{A}$ is closed in X and $\bar{A}^Y = Y \cap \bar{A}$.*

Proposition 2.4. *If $f\colon X \to Y$ is a continuous bijection that is open or closed, then f is a homeomorphism.*

2.2. **Maps and homeomorphisms.**

Proposition 2.5 (Pasting Lemma/sheaf condition)**.** *: Let V be an open subset of a space X and $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ an open cover of V. If $\{f_\alpha\colon U_\alpha \to Y\}$ is a collection of continuous maps to a space Y such that $f_\alpha|_{U_\alpha\cap U_\beta} = f_\beta|_{U_\alpha\cap U_\beta}$ for all $U_\alpha \cap U_\beta \neq \emptyset$ then there exists a unique map $f\colon V \to Y$ such that $f|_{U_\alpha} = f_\alpha$ for all $\alpha \in \mathcal{A}$.*

2.3. **Products, quotients.**

Proposition 2.6 ($\star$)**.** *The canonical projections $p_i\colon \prod_{i=1} X_i \to X_i$ are open maps.*

Proposition 2.7. *If $\prod_{i=1}^N X_i$ is the finite product of the metric spaces $\{(X_i, d_i)\}_{i=1}^N$ equipped with the metric $d^\infty(x, y) = \max_i d_i(x_i, y_i)$, then the topology induced by d^∞ is the product topology.*

Proposition 2.8 ($\star$)**.** *A space X is Hausdorff if and only if the diagonal Δ is closed in $X \times X$ equipped with the product topology.*

Proposition 2.9. *If $d^\alpha, d^\beta = d^\infty$, d^2, or d^1, then $\mathrm{id}\colon \left(\prod_{i=1}^N X_i, d^\alpha\right) \to \left(\prod_{i=1}^N X_i, d^\beta\right)$ is uniformly continuous.*

2.4. **Connectedness.**

Proposition 2.10. *If $f\colon X \to Y$ is a continuous map and X (path) connected, then $f(X)$ is (path) connected.*

Proposition 2.11. *If $A \subseteq X$ is connected and $C \subseteq X$ is open and closed, then $A \subseteq C$ or $A \subseteq X \setminus C$.*

Proposition 2.12 ($\star$)**.** *If A and B are connected and $A \cap B \neq \emptyset$, then $A \cup B$ is connected.*

Proposition 2.13. *If $A \subseteq X$ is a connected subspace, then $\bar{A}$ is a connected subspace.*

Proposition 2.14 ($\star$)**.** *If* $A \subseteq B \subseteq \bar{A} \subseteq X$ *and* A *is connected, then* B *is connected.*

Proposition 2.15. *If* $x \in X$ *and* $C \subseteq X$ *is the connected component of* X *containing* x*, then* C *is the maximal connected set containing* x*.*

Proposition 2.16. *Every open set of a Euclidean space* $\mathbb{R}^n$ *is locally connected and locally path connected.*

Proposition 2.17. X *is locally (path) connected if and only if for any open set* $U \subseteq X$*, the (path) connected components of* U *are open.*

Proposition 2.18. *A path connected space is connected.*

Proposition 2.19. *The connected components of a topological space are closed.*

Proposition 2.20. *If* X *is locally path connected, then the path connected components and connected components coincide.*

2.5. **Compactness.**

Proposition 2.21. *A closed subspace of a compact space is compact.*

Proposition 2.22. *A compact subspace of a Hausdorff space is closed.*

Proposition 2.23. *If* $f\colon X \to Y$ *is a continuous map and* X *is compact, then* $f(X)$ *is compact.*

Proposition 2.24. *If* X *is compact and* $f\colon X \to \mathbb{R}$ *is continuous, then* f *attains its minimum and maximum.*

Proposition 2.25. *If* $f\colon X \to Y$ *is a continuous map between metric spaces and* X *is compact, then* f *is uniformly continuous.*

Proposition 2.26 ($\star$ Lebesgue Covering Lemma)**.** *Let* (X, d) *be a compact metric space. Let* $\mathcal{U} = \{U_\alpha\}$ *be an open cover of* X*. There exists* $\eta > 0$ *such that if* $x, y \in X$ *and* $d(x, y) < \eta$ *then* $\exists U_\alpha \in \mathcal{U}$ *with* $x, y \in U_\alpha$*.*

Proposition 2.27 ($\star$)**.** *A compact metric space is bounded.*

Proposition 2.28. X *is compact iff for every collection of closed subsets* $\{F_\alpha\}_{\alpha \in \mathcal{A}}$ *with the finite intersection property, we have* $\cap_{\alpha \in \mathcal{A}} F_\alpha \neq \emptyset$*.*

Proposition 2.29 ($\star$)**.** *If* $f\colon X \to Y$ *is a continuous map with* X *compact, and* Y *Hausdorff, then* f *is a closed map.*

Proposition 2.30 (Tube Lemma)**.** *Let* X *and* Y *be spaces with* X *compact. Let* $y \in Y$*. If* $\mathcal{W} = \{W_\alpha\}$ *is a family of open sets in* $X \times Y$ *such that* $X \times \{y\} \subseteq \cup_\alpha W_\alpha$*, then there exists a finite open cover* $\mathcal{U}_y = \{U_i\}$ *of* X *and an open neighborhood* $V \in Y$ *of* y *such that each* $U_i \times V$ *is contained in some* $W_\alpha \in \mathcal{W}$*.*

Proposition 2.31. *If* X *and* Y *are compact, then* $X \times Y$ *is compact.*

Proposition 2.32 (Wallace's Theorem)**.** *Let* X *and* Y *be topological spaces with* $A \subseteq X$ *and* $B \subseteq Y$ *compact subspaces. If* $W \subseteq X \times Y$ *is an open set containing* $A \times B$*, then there exists open sets* $U \subseteq X$ *and* $V \subseteq Y$ *such that* $A \times B \subseteq U \times V \subseteq W$*.*

2.6. **Countability and separation properties.**

Proposition 2.33 ($\star$)**.** *If* X *is a second-countable space, then* X *is separable and Lindelöf.*

Proposition 2.34 ($\star$)**.** *A separable metric space is second-countable.*

Proposition 2.35 ($\star$)**.** *A metric space that is compact (or even Lindeĺoff) is second-countable.*

Proposition 2.36. *A subspace of a second-countable space is second countable. The product of two second-countable spaces is second countable.*

Proposition 2.37 ($\star$)**.** *Let* X *be a space. Let* $A \subseteq B \subseteq X$*, with* B *dense in* X*. If* A *is dense in* B*, then* A *is dense in* X*.*

Proposition 2.38 ($\star$)**.** *If X is a separable metric space and $A \subseteq X$ is a subspace, then A is separable.*

Proposition 2.39. *Every metric space is T_4.*

Proposition 2.40. *If X is a compact Hausdorff space, then X is a T_4 space.*

Hint. Use Wallace's Theorem. ☐

2.7. **Local compactness/1-pt. compactification.**

Proposition 2.41. *Let X be locally compact and Hausdorff. If $B \subseteq X$ is a subspace and $B = U \cap A$ with $U \subseteq X$ open and $A \subseteq X$ closed, then B is locally compact.*

Proposition 2.42. *If X is compact Hausdorff and $U \subseteq X$ is an open set, then U is locally compact.*

Proposition 2.43. *If X is locally compact and Hausdorff, then X is T_3.*

2.8. **Complete metric spaces and Baire's Theorem.**

Proposition 2.44 ($\star$)**.** *Any compact metric space is a complete space.*

Proposition 2.45. *Let X be a complete metric space. If a subspace $A \subset X$ is complete, then A is closed. If a subspace $A \subset X$ is a closed, then A is complete.*

Proposition 2.46. *If X,Y are complete metric spaces, then the product $X \times Y$ is a complete metric space.*

Proposition 2.47. *If X is a topological space then* $\mathrm{Bnd}(X) = \{f \colon X \to \mathbb{R} \mid \forall x \in X |f(x)| < \infty\}$ *is a complete metric space with respect to the* sup *norm and the subspace of* $\mathrm{Bnd}(X)$ *consisting of continuous bounded functions is closed.*

Proposition 2.48. *Let (X, d) be a complete metric space. Given $A \subset X$ define*

$$\mathrm{diam}(A) = \sup_{u,v \in A} d(u, v) \in \mathbb{R} \cup \{\infty\}$$

If $\{A_n\}$ is a nested sequence of closed subspaces of X s.t

$$\lim_{n \to \infty} \mathrm{diam}(A_n) = 0,$$

then $\cap_n A_n = \{\mathrm{1pt}\}$.

Proposition 2.49. *A subset $A \subseteq X$ is nowhere dense if and only if* $\mathrm{Int}(\bar{A}) = \emptyset$ *if and only if $X \setminus \bar{A}$ is dense in X.*

Proposition 2.50. *Let X be a space. If $A \subseteq B \subseteq C \subseteq X$ and B is nowhere dense in C, then A is nowhere dense in C.*

Proposition 2.51. *If $A, B \subseteq X$ are nowhere dense in X, then $A \cup B$ is nowhere dense in X.*

Proposition 2.52. *X is a Baire space if and only if the union of any countable collection of closed nowhere dense subsets of X has empty interior.*

Proposition 2.53 (Baire category theorem)**.** *Any complete metric space is a Baire space*

Remark. Prove the above proposition by assuming Theorem $*$ in section 1.8.

Proposition 2.54 ($\star$)**.** *If X is a countable, complete metric space, then X has at least 1 isolated point.*

Proposition 2.55. *If X is a compact metric space such that every point $x \in X$ is a limit point, then X is uncountable.*

3. EXAMPLES, COUNTER-EXAMPLES, AND PATHOLOGIES

(1) A set X equipped with the co-finite topology that is metrizable.
(2) A function on a metric space that is continuous but not uniformly continuous.
(3) A bijective continuous map that is not a homeomorphism.
(4) A bijective, open, and closed function that is not continuous.
(5) An open map that is not closed.
(6) A closed and bounded space that is not compact
(7) A space X and compact subspace A that is not closed
(8) A complete metric space that is homeomorphic to a non-complete space
(9) A meager subspace of $\mathbb{R}$ that is not nowhere dense
(10) A space meager in itself
(11) A connected metric space (X, d) where d is the discrete metric
(12) A non-empty topological space Y such that given any space X and continuous map $f\colon X \to Y$, the image $f(X)$ is connected
(13) A space whose connected components are not open
(14) A connected space that is not path-connected
(15) A connected space that is not locally connected
(16) A subspace of a locally compact space that is not locally compact
(17) An example of a metric space (X, d) where the closure of $N_\epsilon(x) = \{y \in X \mid d(x, y) < \epsilon\}$ is not $D_\epsilon(x) = \{y \in X \mid d(x, y) \leq \epsilon\}$
(18) A space X and connected subspaces $A \subseteq X$ and $B \subseteq X$ such that $A \cap B$ is not connected
(19) A space X, a dense subspace $D \subseteq X$, and a subspace $A \subseteq X$ such that $A \cap D$ is not dense in A
(20) A space X and subspaces $A \subseteq X$ and $B \subseteq X$ such that $\overline{A \cap B} \subsetneq \bar{A} \cap \bar{B}$
(21) Disjoint open sets U and V of a space X whose closures are not disjoint
(22) A separable space that is not Lindelöf*
(23) A Lindelöf space that is not separable*
(24) A separable Lindelöf space that is not second countable*
(25) A function $f\colon \mathbb{R} \to R$ continuous at 1 point
(26) A function $f\colon \mathbb{R} \to \mathbb{R}$ that is neither open nor closed
(27) A uncountable topological space of the first category
(28) A totally disconnected space topological space (X, τ) that is not homeomorphic to X equipped with the discrete topology
(29) Embeddings $f\colon X \hookrightarrow Y$ and $g\colon Y \hookrightarrow X$ such that X is not homeomorphic to Y
(30) Locally connected spaces X and Y and a continuous map $f\colon X \to Y$ such that $f(X)$ is not locally connected*
(31) A complete metric space that is not locally compact

$*$ = brutal

4. REFERENCES

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