Math 171
Solutions to Quiz 1

1. Let $x, y \in \mathbb{Q}$. Consider $\phi(x+y) = \phi(x) * \phi(y)$. Then $ax + ay + b = (ax+b) * (ay+b)$. Observe $ax + ay + b = (ax+b) + (ay+b) - b$. We then can conclude $u * v = u + v - b$.

   OR:
   First compute the inverse of ϕ:

   $$\psi(u) = \phi^{-1}(u) = \frac{u-b}{a}.$$

   Let $u = \phi(x)$ and $v = \phi(y)$. Then

   $$\begin{aligned} u * v =& \phi(x+y) = \phi(\psi(u) + \psi(v)) \\ =& \phi\left(\frac{u-b}{a} + \frac{v-b}{a}\right) \\ =& a\left(\frac{u-b}{a} + \frac{v-b}{a}\right) + b \\ =& u + v - b. \end{aligned}$$

2. Reflexive: If $q \in \mathbb{Q}$ then $q - q = 0 \in \mathbb{Z}$. Hence $q \sim q$.
   Symmetric: If $p \sim q$ then there exists $n \in \mathbb{Z}$ s.t. $p - q = n$. Therefore $q - p = -n \in \mathbb{Z}$. Hence $q \sim p$.
   Transitive: Assume $p \sim q$ and $q \sim r$. Therefore $p - q$ and $q - r$ are integers. Therefore $p - q + q - r = p - r$ is an integer. Hence $p \sim r$.

extra problem just for fun
Let P be an equivalence class for the equivalence relation $\sim$ defined in problem 2. I claim all classes can be uniquely represented by a rational number p/q such that $0 \leq p/q < 1$. I'll show first that every class P contains such a representative. Let p/q be any element of P. If p/q is less than zero, I can always find an integer n s.t. $n + p/q > 0$. Since $n + p/q - p/q = n$, adding n does not change the equivalence class. So we may assume $p/q \geq 0$. If p/q is less than 1, we're done. If not, then $p \geq q$. If $p = q$ then P is the equivalence class containing 1 and therefore 0. So we are done. The last case is when $p > q$. By the Euclidean algorithm (i.e. division in $\mathbb{Z}$) there exists $b \in \mathbb{Z}$ and $r \in \mathbb{Z}$ s.t. $p = bq + r$, and $0 < r < q$. Divide both sides by q to get $p/q = b + r/q$. b is an integer, hence $p/q \sim r/q$ and $0 < r/q < 1$.

I'll leave uniqueness to you as an exercise, i.e. show for every equivalence class P there is a unique representative $p/q \in P$ s.t. $0 \leq p/q < 1$. After you prove this, you can conclude that equivalence classes corresponding to $\sim$ are in a 1 to 1 correspondence with the set $\mathbb{Q} \cap [0, 1)$.