Mathematics 144 11/17/ 2009 Key
Test #2 name

1. 2. 3. 4. 5. 6. 7. TOTAL

PROBLEM 1. Suppose that $\mathcal{I}$ is a family of inductive sets. Prove that $\cap\mathcal{I}$ is an inductive set.

See lecture 10.27.09.
notes

PROBLEM 2: For each prime positive integer k, let $D_k = \{n \in N : k$ divides n, i.e there is a positive integer m such that $n = km\}$. Is $\mathbb{D} = \{D_k : k$ is a prime$\}$ a partition of $\mathbb{N}\backslash\{1\}$? [Recall that 1 is not a prime.] Explain.

A collection $\{P_i\}$ of subsets of X is a partition of X if & only if $X = \bigcup P_i$ & $P_i \cap P_j = \phi$ if $i \neq j$.

Here our collection is $\{D_K\} = \mathbb{D}$. It is true that $\mathbb{N}\backslash\{1\} = \bigcup_k D_K$. But it is not true that $D_K \cap D_\ell = \phi$ if $k \neq \ell$.

Example: Consider D_2 & D_3. $2|6 \therefore 6 \in D_2$ & $3|6 \therefore 6 \in D_3$

$\therefore D_2 \cap D_3 \neq \phi$. $\therefore \mathbb{D}$ is not a partition of $\mathbb{N}\backslash\{1\}$.

PROBLEM 3: Define a relation on $\mathbb{N}\times\mathbb{N}$ by $(a,b)\sim(x,y)\Leftrightarrow ay=bx$. $\sim$ is reflexive and symmetric. (i.) Using the arithmetic of $\mathbb{N}$, prove that $\sim$ is an equivalence relation by showing that it is transitive.(ii.) The collection of all $\sim$ equivalence classes, denoted by $(\mathbb{N}\times\mathbb{N})/\sim$ [and often referred to as $N\times N$ modulo $\sim$] will be defined to be $\mathbb{Q}^+$. Let $\gamma:\mathbb{N}\times\mathbb{N}\to\mathbb{Q}^+$ be the natural, or canonical, map from $N\times N$ onto $\mathbb{Q}^+$. Suppose the $\frac{p}{q}\in\mathbb{Q}^+$. Is $\gamma^{-1}(\frac{p}{q})$ finite or infinite? Explain.

(i) see Solutions to quiz #2.

(ii) Let $\gamma:\mathbb{N}\times\mathbb{N}\to\mathbb{Q}^+$ be the map $\gamma((p,q))=\frac{p}{q}$. Then $\gamma^{-1}(p/q)=\{(np,nq)\mid n\in\mathbb{N}\}$ since $\frac{np}{nq}=p/q$. $\therefore\ \gamma^{-1}(p/q)$ is not finite.

PROBLEM 4. Prove or give a counter-example to the statement: $f^{-1}(A\cap B)\subseteq f^{-1}(A)\cap f^{-1}(B)$.

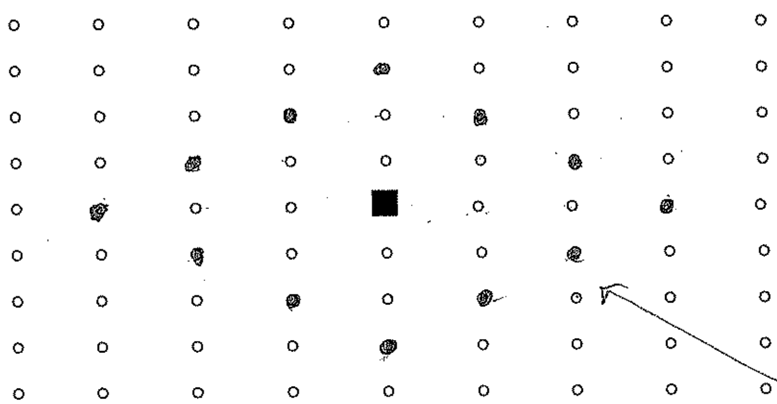
$A\cap B\subseteq A\Rightarrow f^{-1}(A\cap B)\subseteq f^{-1}(A)$

$A\cap B\subseteq B\Rightarrow f^{-1}(A\cap B)\subseteq f^{-1}(B)$

$\therefore\ f^{-1}(A\cap B)\subseteq f^{-1}(A)\cap f^{-1}(B)$ $\square$

PROBLEM 5. Define a function $sumab : \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}_0$ by $sumab((m,n)) = |m| + |n|$. Note that $sumab^{-1}(0) = \{(0,0)\}$ and $sumab^{-1}(1) = \{(1,0),(0,1),(-1,0),(0,1)\}$.

(*i.*) Find $sumab^{-1}(3)$ on the chart below. ■ is the origin.



$sumab^{-1}(3) = \{(m,n) \mid |m|+|n|=3\}$

$= \{(3,0),(0,3),(-3,0),(0,-3),(2,1),(1,2),(-2,1),(1,-2),(2,-1),(-1,2),(-2,-1),(-1,-2)\}$

(*ii.*) How many elements does $sumab^{-1}(n)$ have? Explain.

$sumab^{-1}(0) = \{0,0\}$ ∴ 1 elt. Look at the size of the "diamond"

$|sumab^{-1}(0)|, |sumab^{-1}(1)|, |sumab^{-1}(2)|, \ldots$ (= 1, = 4, = 8)

∴ $|sumab^{-1}(0)| = 1$

$|sumab^{-1}(n)| = 4n$, $n>0$.

PROBLEM 6. Explain why $Fin2^N$ is countable.

$Fin\,2^{\mathbb{N}} = \{$ set of all finite sequences of 1's : 0's $\}$

Let $\phi : Fin\,2^{\mathbb{N}} \to \mathbb{N}$ be the function.

$\phi(\{n_0, \ldots, n_k\}) = n_0 2^0 + n_1 2^1 + n_2 2^2 + \ldots + n_k 2^k$

This function has an inverse $\phi^{-1} : \mathbb{N} \to Fin\,2^{\mathbb{N}}$

which takes a natural number to its unique binary representation. ∴ $\phi : Fin\,2^{\mathbb{N}} \to \mathbb{N}$ is a bijection

Hence $Fin\,2^{\mathbb{N}}$ is countable.

Problem #17: Given bijections $\alpha: \mathbb{N} \xrightarrow{\sim} \mathbb{N} \times \mathbb{N}$,

$\beta: \mathbb{N} \times \mathbb{N} \xrightarrow{\sim} \mathbb{Q}^+$, $\xi: \mathbb{N} \to \mathbb{Z}$.

(i) bijection from $\mathbb{N}$ to $\mathbb{Z} \times \mathbb{Z}$:

Let $\psi: \mathbb{N} \times \mathbb{N} \to \mathbb{Z} \times \mathbb{Z}$ be the function $\psi((m,n)) = (\xi(m), \xi(n))$. Then ψ is a bijection since ξ is a bijection.

$\therefore$ we have $\mathbb{N} \xrightarrow{\alpha} \mathbb{N} \times \mathbb{N} \xrightarrow{\psi} \mathbb{Z} \times \mathbb{Z}$

The composition $\psi \circ \alpha : \mathbb{N} \to \mathbb{Z} \times \mathbb{Z}$ is a bijection since α & ψ are bijections.

(ii) bijection from $\mathbb{N}$ to $\mathbb{Q}$:

Note that: $\mathbb{N} \xrightarrow{\alpha} \mathbb{N} \times \mathbb{N} \xrightarrow{\beta} \mathbb{Q}^+$ gives a bijection $\beta \circ \alpha : \mathbb{N} \to \mathbb{Q}^+$.

Note that: $\mathbb{Q} = \mathbb{Q}^+ \cup \{0\} \cup \mathbb{Q}^-$ & $\mathbb{N} = \{z \in \mathbb{Z} \mid z > 0\}$
& $\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \mathbb{N}^-$ & $\mathbb{N}^- = \{z \in \mathbb{Z} \mid z < 0\}$.

Let $\phi: \mathbb{Z} \to \mathbb{Q}$ be the function $\phi(z) = \begin{cases} 0 & \text{if } z = 0 \\ \beta \circ \alpha(z) & \text{if } z > 0 \\ -\beta \circ \alpha(-z) & \text{if } z < 0 \end{cases}$

Then ϕ is a bijection since $\beta \circ \alpha$ is a bijection.

We have $\mathbb{N} \underset{\text{bijection}}{\xrightarrow{\xi}} \mathbb{Z} \xrightarrow{\phi} \mathbb{Q}$ $\therefore$ $\phi \circ \xi : \mathbb{N} \to \mathbb{Q}$ is a bijection