PROBLEM 1. Suppose that \( I \) is a family of inductive sets. Prove that \( \bigcap I \) is an inductive set.

See lecture 10.27.09.

PROBLEM 2: For each prime positive integer \( k \), let \( D_k = \{ n \in \mathbb{N} : k \text{ divides } n \} \), i.e there is a positive integer \( m \) such that \( n = km \). Is \( \mathbb{D} = \{ D_k : k \text{ is a prime} \} \) a partition of \( \mathbb{N}\backslash\{1\} \)? [Recall that 1 is not a prime.] Explain.

A collection \( \{ P_i \} \) of subsets of \( X \) is a partition of \( X \) if and only if \( X = \bigcup P_i \) and \( P_i \cap P_j = \emptyset \) if \( i \neq j \).

Here our collection is \( \{ D_k \} = \mathbb{D} \). It is true that \( \mathbb{N}\backslash\{1\} = \bigcup_{k=2}^{\infty} D_k \). But it is not true that \( D_k \cap D_l = \emptyset \) if \( k \neq l \).

Example: Consider \( D_2 \cap D_3 \). \[ 2 \mid 6 \implies 6 \in D_2 \land 3 \mid 6 \implies 6 \in D_3 \]
\[ \therefore D_2 \cap D_3 \neq \emptyset \]
\[ \therefore \mathbb{D} \text{ is not a partition of } \mathbb{N}\backslash\{1\} \]
PROBLEM 3: Define a relation on \( \mathbb{N} \times \mathbb{N} \) by \((a, b) \sim (x, y) \iff ay = bx. \sim \) is reflexive and symmetric. (i.) Using the arithmetic of \( \mathbb{N} \), prove that \( \sim \) is an equivalence relation by showing that it is transitive.(ii.) The collection of all \( \sim \) equivalence classes, denoted by \((\mathbb{N} \times \mathbb{N}) / \sim \) [and often referred to as \( N \times N \) modulo \( \sim \)] will be defined to be \( \mathbb{Q}^+ \). Let \( \gamma : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{Q}^+ \) be the natural, or canonical, map from \( N \times N \) onto \( \mathbb{Q}^+ \). Suppose the \( \frac{p}{q} \in \mathbb{Q}^+ \). Is \( \gamma^{-1}(\frac{p}{q}) \) finite or infinite? Explain.

(i) see solutions to Quiz #2.

(ii) Let \( \gamma : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{Q}^+ \) be the map
\[
\gamma((p,q)) = \frac{p}{q}, \quad \text{Then } \gamma^{-1}(\frac{p}{q}) = \left\{ (np,nq) \mid n \in \mathbb{N} \right\}
\]

since \( \frac{np}{nq} = \frac{p}{q} \). \( \gamma^{-1}(\frac{p}{q}) \) is not finite.

PROBLEM 4. Prove or give a counter-example to the statement:
\( f^{-1}(A \cap B) \subseteq f^{-1}(A) \cap f^{-1}(B) \).

\[
\begin{align*}
A \cap B \subseteq A & \Rightarrow f^{-1}(A \cap B) \subseteq f^{-1}(A) \\
A \cap B \subseteq B & \Rightarrow f^{-1}(A \cap B) \subseteq f^{-1}(B) \\
\therefore f^{-1}(A \cap B) & \subseteq f^{-1}(A) \cap f^{-1}(B)
\end{align*}
\]
PROBLEM 5. Define a function $\text{sumab} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}_0$ by $\text{sumab}((m,n)) = |m| + |n|$. Note that $\text{sumab}^{-1}(0) = \{(0,0)\}$ and $\text{sumab}^{-1}(1) = \{(1,0), (0,1), (-1,0), (0,1)\}$.

(i.) Find $\text{sumab}^{-1}(3)$ on the chart below. ■ is the origin.

$\text{sumab}^{-1}(3) = \{(m,n) \mid |m| + |n| = 3\}$

$= \{(3,0), (0,3), (-3,0), (0,3), (2,1), (1,2), (-2,1), (1,2), (2,1)\}$

(ii.) How many elements does $\text{sumab}^{-1}(n)$ have? Explain.

$\text{sumab}^{-1}(0) = \{(0,0)\} = 1 \text{ elt. }$. Look at the size of the "diamond".

$|\text{sumab}^{-1}(0)| = 1$

$|\text{sumab}^{-1}(n)| = 4n, n > 0$.

PROBLEM 6. Explain why $\text{Fin}_2^\mathbb{N}$ is countable.

Let $\phi : \text{Fin}_2^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ be the function,

$\phi (\{n_0, \ldots, n_k\}) = n_0 2^0 + n_1 2^1 + n_2 2^2 + \ldots + n_k 2^k$.

This function has an inverse $\phi^{-1} : \mathbb{N}^\mathbb{N} \to \text{Fin}_2^\mathbb{N}$ which takes a natural number to its unique binary representation. $\phi : \text{Fin}_2^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is a bijection.

Hence $\text{Fin}_2^\mathbb{N}$ is countable.
Problem 7: Given bijections \( \alpha : \mathbb{N} \to \mathbb{N} \times \mathbb{N} \), \( \beta : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}^+ \), \( \gamma : \mathbb{N} \to \mathbb{Z} \).

(ii) bijection from \( \mathbb{N} \) to \( \mathbb{Z} \times \mathbb{Z} \):

Let \( \psi : \mathbb{N} \times \mathbb{N} \to \mathbb{Z} \times \mathbb{Z} \) be the function

\[
\psi(m,n) = (\gamma(m), \gamma(n)).
\]

Then \( \psi \) is a bijection since \( \gamma \) is a bijection.

We have \( \mathbb{N} \xrightarrow{\alpha} \mathbb{N} \times \mathbb{N} \xrightarrow{\psi} \mathbb{Z} \times \mathbb{Z} \).

The composition \( \psi \circ \alpha : \mathbb{N} \to \mathbb{Z} \times \mathbb{Z} \) is a bijection since \( \alpha \) and \( \psi \) are bijections.

(ii) bijection from \( \mathbb{N} \) to \( \mathbb{Q} \):

Note that \( \mathbb{N} \mathcal{I} \xrightarrow{\alpha} \mathbb{N} \times \mathbb{N} \xrightarrow{\beta} \mathbb{Q}^+ \) gives a bijection \( \beta \circ \alpha : \mathbb{N} \to \mathbb{Q}^+ \).

Note that \( \mathbb{Q} = \mathbb{Q}^+ \cup \{0\} \cup \mathbb{Q}^- \), \( \mathbb{N} = \{ z \in \mathbb{Z} \mid z > 0 \} \), \( \mathbb{N}^- = \{ z \in \mathbb{Z} \mid z < 0 \} \).

Let \( \phi : \mathbb{Z} \to \mathbb{Q} \) be the function \( \phi(z) = \begin{cases} 
0 & \text{if } z = 0 \\
\beta \circ \alpha(z) & \text{if } z > 0 \\
-\beta \circ \alpha(z) & \text{if } z < 0
\end{cases} \).

Then \( \phi \) is a bijection since \( \beta \circ \alpha \) is a bijection.

We have \( \mathbb{N} \xrightarrow{\gamma} \mathbb{Z} \xrightarrow{\phi} \mathbb{Q} \), \( \phi \circ \gamma : \mathbb{N} \to \mathbb{Q} \) is a bijection.