#1] Let A be a set of inductive sets (non-empty).

Claim: $\bigcap A = \{x \mid x \in S \ \forall S \in A\}$ is inductive.

pf] $\phi \in S \ \forall S \in A$ since every set in A is inductive. $\therefore \phi \in \bigcap A$. Assume $n \in \bigcap A$. Then $n \in S \ \forall S \in A$. Every set in A is inductive $\therefore n^+ \in S \ \forall S \in A$. $\therefore n^+ \in \bigcap A$.

$\therefore \bigcap A$ is inductive $\square$.

#2] Comp & Decomp problems: easy.

#3] Comp & Decomp problems: easy.

#4] Show $\mathbb{Z}$ is countably infinite:

A set is countably infinite if and only if it can be written as a sequence w/o repetition.

$\mathbb{Z} = (0, 1, -1, 2, -2, 3, -3, \ldots, n, -n, \ldots)$

OR: Consider the function $f: \mathbb{N} \to \mathbb{Z}$, $f(n) = \begin{cases} -\left(\frac{n-1}{2}\right), & n \text{ odd} \\ \frac{n}{2}, & n \text{ even.} \end{cases}$

It has an inverse $f^{-1}(z) = \begin{cases} 2z, & z > 0 \\ 1-2z, & z \leq 0 \end{cases}$

$\therefore f$ is a bijection.

#5] Describe a bijection from $\mathbb{N}$ onto $\mathbb{N} \times \mathbb{N}$:

$\mathbb{N} = (1, 2, 3, 4, 5, \ldots$

$\mathbb{N} \times \mathbb{N} = ((1,1), (2,1), (1,2), (3,1), (2,2), (1,3), \ldots$

②

#6 (i). $\mathbb{P}$ is partition of X. $\mathbb{P}$ is countably infinite (given) $\therefore \mathbb{P} = \{P_1, P_2, P_3, \ldots\}$. Each P_i is countably infinite (given). $\therefore$ write each P_i as:

$$P_i = \{x_{i1}, x_{i2}, x_{i3}, \ldots\}, \text{ with } x_{ij} \text{ distinct.}$$

$\mathbb{P}$ partition of X $\therefore$ $X = \bigcup \mathbb{P}$ & $P_i \cap P_j = \phi$ if $i \neq j$.

$\therefore$ if $x \in X$, then $x \in \bigcup \mathbb{P} \Rightarrow \exists\, i$ s.t. $x \in P_i$ (and this i is unique). $P_i = \{x_{i1}, x_{i2}, \ldots\}$ $\therefore$ $x = x_{ij}$ for some j.

$\therefore$ define $\theta : X \to \mathbb{N} \times \mathbb{N}$ by $\theta(x) = (i, j)$ where $x = x_{ij}$.

It is 1-1 since each x is uniquely specified as x_{ij}.

If $(n, m) \in \mathbb{N} \times \mathbb{N}$, then $\exists P_n \in \mathbb{P}$ since $\mathbb{P}$ is countably infinite & $x_{nm} \in P_n$ since P_n countably infinite

$\therefore$ $\theta(x_{nm}) = (n, m)$ $\therefore$ θ onto. $\therefore$ θ bijection.

(ii) $\mathbb{P}$ countable $\Rightarrow \mathbb{P} = \{P_1, P_2, \ldots\}$ (may have repeats) Countable $\therefore$ $P_i = \{x_{i1}, x_{i2}, x_{i3}, \ldots\}$ (may have repeats.

Then since $X = \bigcup \mathbb{P}$, write X as the sequence (with possible repeats):

$$X = (x_{11}, x_{12}, \ldots, x_{21}, x_{22}, \ldots\ x_{31}, x_{32}, \ldots\ )$$

(iii) $f: X \to \mathbb{N}$ s.t. $f^{-1}(n)$ is finite, $\forall n \in \mathbb{N}$.

Let $\mathbb{P} = \{f^{-1}(1), f^{-1}(2), f^{-1}(3), \dots\}$.

If $x \in X$ then $f(x) = n \in \mathbb{N}$. $\therefore x \in f^{-1}(n)$.

$\therefore X = \bigcup \mathbb{P}$. If $x \in f^{-1}(n) \cap f^{-1}(m)$ then

$f(x) = n$ & $f(x) = m$. f is a function $\therefore n = m$.

$\therefore f^{-1}(n) \cap f^{-1}(m) = \phi$ if $n \neq m$. $\therefore \mathbb{P}$ is a

partition. $\mathbb{P}$ is countable & ~~each~~ $f^{-1}(n)$ is finite $\therefore$

countable. $\therefore$ by part (ii) X is countable.

(iv) Let $f: X \to \mathbb{N}$ be <u>onto</u> s.t. $f^{-1}(n)$ is countable, $\forall n \in \mathbb{N}$.

Let $\mathbb{P} = \{f^{-1}(1), f^{-1}(2), \dots\}$ as in part (iii).

From part (iii) $\mathbb{P}$ is a partition. Every $f^{-1}(n)$ is

countable. $\therefore$ by part (ii) $X = \bigcup \mathbb{P}$ is countable.

Since f is <u>onto</u> $f^{-1}(n) \neq \phi$ $\forall n \in \mathbb{N}$.

$\therefore |f^{-1}(n)| \neq 0$ $\therefore$ X is not just countable

but countably infinite.

#7] Claim: $\mathbb{Q}^+ \approx \mathbb{N}$

pf] From the definition of $(m,n) \sim (p,q)$ written above the problem in the handout we have a function:

$\eta: \mathbb{N} \times \mathbb{N} \to \mathbb{Q}^+$ which is onto. From problem #5 we have a bijection $\theta: \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ $\therefore$ the map $\eta \circ \theta: \mathbb{N} \to \mathbb{Q}^+$ is <u>onto</u>. $\therefore$ By remark 22, there is an injective map →

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$\phi: \mathbb{Q}^+ \to \mathbb{N}$. We also have an injective map $i: \mathbb{N} \hookrightarrow \mathbb{Q}^+$ which is the inclusion of $\mathbb{N}$ into $\mathbb{Q}^+$. $\therefore$ By Schroder-Bernstein $\exists$ bijection $\psi: \mathbb{N} \to \mathbb{Q}^+$ $\therefore$ $\mathbb{N} \simeq \mathbb{Q}^+$.

#8] Let $S(n) = \{\bar{a} \in \mathbb{W} \mid a_n = 1 \text{ \& } a_i = 0 \ \forall i > n\}$

(example: $S(2) = \{(0,1,0 \to), (1,1,0,\to)\}$

$S(1) = \{(1,0,\to)\}$, $S(0) = \{(0,\to)\}$

Let $\mathbb{P}$ be the family of sets $\mathbb{P} = \{S(0), S(1), S(2), \ldots\}$

Then $\mathbb{W} = \bigcup \mathbb{P}$ since every sequence $\bar{a} \in \mathbb{W}$ has finite # of one's & $\therefore$ has a "last one".

If $\bar{a} \in S(n) \cap S(m)$, then $a_n = 1$ & $a_i = 0$ $i > n$ & $a_m = 1$ & $a_j = 0$ $j > m$. $\therefore$ $n = m$.

$\therefore$ $\mathbb{P}$ is a partition of $\mathbb{W}$ & each $S(n)$ is countable (in fact $|S(n)| = 2^{n-1}$) $\therefore$ by problem #6 (ii) $\mathbb{W}$ is countable.

#9] Let $\mathbb{K}$ be the set of all finite sequences of zero's & one's.

Let $\phi: \mathbb{K} \to \mathbb{N}$ be the function

$$\phi((a_0, a_2, \ldots, a_n)) = a_0 2^0 + a_1 2^1 + \ldots + a_n 2^n$$

Since every $n \in \mathbb{N}$ has a unique binary decomposition, ϕ is a bijection $\therefore$ $\mathbb{K}$ countable.

#10 Let P be the set of all polys. with rational coefficients. Each $p \in P$ can be represented by a unique finite sequence of rationals:

$$(a_0, a_1, \ldots, a_n) \rightsquigarrow a_0 + a_1 x + \ldots + a_n x^n, \quad a_n \neq 0.$$

$\mathbb{Q}$ is countable $\therefore \exists$ bijection $\phi : \mathbb{Q} \to \mathbb{N}$.

Let $\times\phi : P \to FSIN$ be the function

$$\times\phi(a_0 + a_1 x + \ldots + a_n x^n) = (\phi(a_0), \phi(a_1), \ldots, \phi(a_n))$$

$\times\phi$ is a bijection since ϕ is a bijection & FSIN is countable $\therefore P$ countable.

#11 (i) Let $\eta : 2^{\mathbb{N}} \to \mathbb{R}$ be the map.

$$\eta(a_1, a_2, \ldots) = 0.a_1 a_2 \ldots$$

Assume $\eta(a_1, a_2, a_3, \ldots) = \eta(b_1, b_2, \ldots)$

Then $0.a_1 a_2 \ldots = 0.b_1 b_2 b_3 \ldots$

Since a_i's & b_i's are either 0 or 1, this implies

$a_1 = b_1, a_2 = b_2, \ldots \quad \therefore (a_1, a_2, \ldots) = (b_1, b_2, \ldots)$

$\therefore \eta$ is 1-1.

(ii) $\eta : 10^{\mathbb{N}} \to \mathbb{R}$ not 1-1:

Consider $\bar{a} = (1, 0, \to)$ & $\bar{b} = (0, 9, 9, \to)$

Then $\bar{a} \neq \bar{b}$ but $\eta(a_1, a_2, \ldots) = 0.1 = 0.0999\ldots = \eta(b_1, b_2, \ldots)$

#12] $2^{\mathbb{N}}$ not countable. Use Cantor diagonal argument. See my notes from 10.13.09 (online)

#13] Let X be countable; $A \subseteq X$. X countable $\Rightarrow \exists$ bijection $\phi: X \to \mathbb{N}$. Let $\phi_A : A \to \mathbb{N}$ be the function ϕ restricted to A. i.e. $\phi_A(a) = \phi(a)$ $\forall a \in A$. Since ϕ is a bijection, ϕ is 1-1 i.e. ϕ_A is 1-1. By remark 22, A is countable.

#14] Claim: $\mathbb{R}$ not countable

pf] Assume $\mathbb{R}$ is countable. Then by problem #13 every subset of $\mathbb{R}$ is countable. By problem #11 i the map $\eta: 2^{\mathbb{N}} \to \mathbb{R}$ is 1-1.

The image of η is $\eta(2^{\mathbb{N}}) = \{\eta(\bar{a}) \mid \bar{a} \in 2^{\mathbb{N}}\}$

Since $\eta(2^{\mathbb{N}}) \subseteq \mathbb{R}$, $\eta(2^{\mathbb{N}})$ is countable.

Since η is 1-1, it is a bijection onto its image $\therefore$ $2^{\mathbb{N}}$ is countable. This contradicts solution to problem #12. $\therefore$ $\mathbb{R}$ cannot be countable.

#15] claim: $\exists$ bijection from $2^{\mathbb{N}}$ to $7^{\mathbb{N}}$.

Pf] Since $2=\{0,1\}$ & $7=\{0,1,2,3,4,5,6\}$.
$2 \subseteq 7$. $\therefore$ any sequence in $2^{\mathbb{N}}$ is a sequence in $7^{\mathbb{N}}$ $\therefore$ $\exists$ 1-1 function $i: 2^{\mathbb{N}} \to 7^{\mathbb{N}}$ given by $i(\bar{a}) = \bar{a}$.

Let $\beta: 7 \to 2\times 2\times 2$ be the map that takes $n\in 7$ to its unique binary rep representation i.e. $\beta(0)=(0,0,0)$, $\beta(1)=(1,0,0)$, $\beta(2)=(0,1,0)$, $\beta(3)=(1,1,0)$, ...

Then define $\gamma: 7^{\mathbb{N}} \to 2^{\mathbb{N}}$ to be the function $\gamma((b_1,b_2,b_3,\dots)) = (\beta(b_1), \beta(b_2), \beta(b_3), \dots)$

Then γ is 1-1 since binary representation is unique.

$\therefore$ by Schroeder-Bernstein Thm $\exists$ bijection from $2^{\mathbb{N}}$ to $7^{\mathbb{N}}$.

#29] claim: $A\neq B$ then $A\setminus B \neq \emptyset$ or $B\setminus A \neq \emptyset$.

Pf] If $A\neq B$ then $A \not\subseteq B$ or $B \not\subseteq A$.
i.e. $\exists\, a\in A$ s.t. $a\notin B$ or $\exists\, b\in B$ s.t. $b\notin A$.
$\therefore$ $a\in A\setminus B$ or $b\in B\setminus A$ $\therefore$
$A\setminus B\neq\emptyset$ or $B\setminus A\neq\emptyset$.

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#31b] Claim: if $\mathcal{A}$ collection of sets & $A \in \mathcal{A}$ then $\bigcap \mathcal{A} \subseteq A \subseteq \bigcup \mathcal{A}$.

pf] Let $x \in \bigcap \mathcal{A}$. Then $x \in A' \ \forall A' \in \mathcal{A}$

$A \in \mathcal{A} \therefore x \in A \therefore \bigcap \mathcal{A} \subseteq A$. Now

Let $x \in A$. By def $\bigcup \mathcal{A} = \{x \mid \exists A' \in \mathcal{A} \text{ s.t. } x \in A'\}$

Since $A \in \mathcal{A}$, $x \in \bigcup \mathcal{A} \therefore A \subseteq \bigcup \mathcal{A}$.

#33c] Let $(X, \leq)$ is partially ordered set.

- Assume A is left-set: Let $a \in A$. By def of left set $\downarrow a \subseteq A$.

$\downarrow a = \{x \mid x \leq a\}$ by def. $\therefore$ If $x \leq a$, $x \in A$.

- Assume whenever $a \in A$ & $x \leq a$, $x \in A$:

Let $\downarrow A = \{x \in X \mid x \leq a \text{ for some } a \in A\}$.

WTS $A = \downarrow A$. Let $a \in A$. Then $a \leq a \Rightarrow a \in \downarrow A$

$\therefore A \subseteq \downarrow A$. Let $x \in \downarrow A$. Then $\exists a \in A$ s.t. $x \leq a$.

By our assumption $x \leq a \Rightarrow x \in A \therefore \downarrow A \subseteq A \therefore$

$A = \downarrow A$.

- Assume $A = \downarrow A$: WTS $A = \bigcup \{\downarrow y \mid y \in A\}$

Let $a \in A$. Then $a \in \downarrow a \Rightarrow a \in \bigcup\{\downarrow y \mid y \in A\}$

$\therefore A \subseteq \bigcup\{\downarrow y \mid y \in A\}$

Let $x \in \bigcup\{\downarrow y \mid y \in A\}$. $\therefore \exists \downarrow y$ s.t. $x \in \downarrow y$.
$\Rightarrow x \leq y$. Since $y \in A$, $x \in \downarrow A$ by def of $\downarrow A$.
$\therefore x \in A$ by our assumption $\therefore \bigcup\{\downarrow y \mid y \in A\} \subseteq A$

$\therefore A = \bigcup\{\downarrow y \mid y \in A\}$.

- Assume $A = \bigcup\{\downarrow y \mid y \in A\}$. we WTS A is a left set. Let $a \in A$. Let $x \in \downarrow a$. Since $A = \{x \mid \exists y \in A$ s.t. $x \in \downarrow y\}$.
$\therefore a \in A$, we have $x \in A$ $\therefore \downarrow a \subseteq A$ $\therefore$ A is left set.

#34] (i) Assume $p \leq q$. Let $x \in \downarrow p$. Then $x \leq p$ by def of $\downarrow p$. $\therefore x \leq q \Rightarrow x \in \downarrow q \Rightarrow \downarrow p \subseteq \downarrow q$.

(ii) Assume $p < q$. Let $x \in \downarrow p$. Then $x \leq p < q$
$\Rightarrow x < q$ $\therefore x \neq q$ & $x \in \downarrow q$. $\therefore x \in \downarrow q \setminus \{q\}$.
$\therefore \downarrow p \subseteq \downarrow q \setminus \{q\} \subset \downarrow q \Rightarrow \downarrow p \subset \downarrow q$.

(iii) WTS $\downarrow q \setminus \{q\}$ is a left set: $\therefore$

Let $x \in \downarrow q \setminus \{q\}$. WTS $\downarrow x \subseteq \downarrow q \setminus \{q\}$.
Let $y \in \downarrow x$. Then $y \leq x < q \Rightarrow y < q \Rightarrow$
$y \in \downarrow q \setminus \{q\}$ $\therefore \downarrow x \subseteq \downarrow q \setminus \{q\}$ $\therefore$
$\downarrow q \setminus \{q\}$ is a left set.

(10)

#41(f) Let $L(\mathbb{Q})$ the set of left sets of $\mathbb{Q}$.

Let $A \subseteq L(\mathbb{Q})$. Then A is a set of left sets of $\mathbb{Q}$

(i.e. for all $S \in A$, S is a left set of $\mathbb{Q}$).

WTS $\bigcap A = \{x \mid x \in S \ \forall S \in A\}$ is a left set:

Let $x \in \bigcap A$. Then $\forall S \in A$ $x \in A$. $\therefore \downarrow x \subseteq A$.

$\forall A \in S$. $\therefore \downarrow x \subseteq \bigcap A$ $\therefore \bigcap A$ is a left set.

WTS $\bigcup A = \{x \mid \exists S \in A \text{ s.t. } x \in S\}$ is a left set.

Let $x \in \bigcup A$. Then $x \in S$ for some $S \in A$.

S a left set $\Rightarrow$ $\downarrow x \subseteq S$. $\therefore$ $\downarrow x \subseteq \bigcup A$.

$\therefore$ $\bigcup A$ is a left set.

#42 g. Claim: $L(\mathbb{Q})$ is linearly ordered.

pf Let $A, B \in L(\mathbb{Q})$ WTS $A \subseteq B$ or $B \subseteq A$.

Assume $A \not\subseteq B$. $\therefore \exists a \in A$ s.t. $a \notin B$.

$\therefore \forall b \in B$ $a \not\leq b$, by #33c $\therefore$ $\forall b \in B$ $b \leq a$ since $\mathbb{Q}$ is linearly ordered

$\therefore B \subseteq \downarrow a$. A left set $\Rightarrow$ $\downarrow a \subseteq A$ $\therefore$ $B \subseteq A$

$\therefore$ $L(\mathbb{Q})$ is linearly ordered.

42h $L(\mathbb{Q})$ is complete:

WTS that every nonempty subset of $L(\mathbb{Q})$ has a l.u.b. & a g.l.b.:

let $A \subset L(\mathbb{Q})$ be a set of left sets of $\mathbb{Q}$.

Clearly $\forall S \in A$, $S \subseteq \bigcup A$ by problem #31b & by problem #41f, $\bigcup A \in L(\mathbb{Q})$. $\therefore \bigcup A$ is a upper bound of A. Suppose $\forall S \in A$, $S \subseteq B$, for some $B \in L(\mathbb{Q})$. Let $x \in \bigcup A$. Then $\exists S \in A$ s.t. $x \in S$. $S \subseteq B \Rightarrow x \in B$ $\therefore \bigcup A \subseteq B$. $\therefore \bigcup A$ is the <u>least</u> upper bound.

Clearly $\bigcap A \subseteq S$ $\forall S \in A$ by problem #31b. & by problem #41f $\bigcap A \in L(\mathbb{Q})$. Suppose $B \subseteq S$ $\forall S \in A$. let $x \in B$. Then $\forall S$ $B \subseteq S \Rightarrow x \in S$. Since $\bigcap A = \{x \mid x \in S \ \forall S \in A\}$, we have $x \in \bigcap A$. $\therefore B \subseteq \bigcap A \Rightarrow \bigcap A$ is the greatest lower bound.

$\therefore L(\mathbb{Q})$ complete.

(12)

#44 j] let $i: \mathbb{Q} \to L(\mathbb{Q})$ be the map
$i(q) = \downarrow q \setminus \{q\}$.

Claim i is 1-1:

Pf] Assume $i(q) = i(p)$ $\therefore$ $\downarrow p \setminus \{p\} = \downarrow q \setminus \{q\}$.

$q \notin \downarrow q \setminus \{q\} = \downarrow p \setminus \{p\} \Rightarrow q \nless p \Rightarrow p \leq q$.

$p \notin \downarrow p \setminus \{p\} = \downarrow q \setminus \{q\} \Rightarrow p \nless q \Rightarrow q \leq p$

$\therefore p = q$ $\therefore$ i is 1-1 □

#45 k] let $L^*(\mathbb{Q}) = L(\mathbb{Q}) \setminus \{\downarrow p \mid p \in \mathbb{Q}\}$.

Claim $L^*(\mathbb{Q})$ has least upper bound property

Pf] let $A \subseteq L^*(\mathbb{Q})$ be non-empty subset of left sets

$\therefore$ $\exists B \in L^*(\mathbb{Q})$ s.t. $S \subseteq B$ $\forall S \in A$.

WTS A has l.u.b:

Consider $\cup A = \{x \mid \exists S \in A \text{ s.t. } x \in S\}$.

$\forall S \in A$, $S \subseteq \cup A$; $\cup A \subseteq B$ as sets in $L(\mathbb{Q})$. Need to check that $\cup A \in L^*(\mathbb{Q})$ i.e. that $\cup A \neq \downarrow p$ for some $p \in \mathbb{Q}$.

Suppose $\cup A = \downarrow p$. Then $p \in \downarrow p \Rightarrow \exists S \in A$ s.t. $p \in S$. Since S is a left set, $\downarrow p \subseteq S$. Since $S \subseteq \cup A$, $S \subseteq \downarrow p$ $\therefore$ $S = \downarrow p$. This is a contradiction since $S \in L^*(\mathbb{Q})$ $\therefore$

$\bigcup A \neq \downarrow P \Rightarrow \bigcup A \in L^*(\mathbb{Q})$ $\therefore$ $\bigcup A$ is a l.u.b.

<u>Problem:</u> $A, B \in L^*(\mathbb{Q})$ with $A \prec B$. Claim $\exists\, q \in \mathbb{Q}$ s.t. $A \subseteq \downarrow q \setminus \{q\} \prec \downarrow q \subseteq B$.

Pf] If $A < B$ then $\exists\, q \in B$ s.t. $q \notin A$. Let $a \in A$. If $a \notin \downarrow q \setminus \{q\}$ then $a \nless q$ $\Rightarrow q \leq a$. Since A is left set $q \leq a \Rightarrow q \in A$ (by problem #33c). This is a contradiction. $\therefore a \in \downarrow q \setminus \{q\} \Rightarrow A \subseteq \downarrow q \setminus \{q\}$. No left set can be between $\downarrow q \setminus \{q\}$ & $\downarrow q$ $\therefore$ we have $A \subseteq \downarrow q \setminus \{q\} \prec \downarrow q$. Since $q \in B$ & B is a left set, $\downarrow q \subseteq B$ $\therefore$ we have $A \subseteq \downarrow q \setminus \{q\} \prec \downarrow q \subseteq B$.

<u>Problem:</u> Discuss covering pairs in $\mathrm{lex}(2^{\mathbb{N}})$

$\bar{a} \prec \bar{b} \Leftrightarrow \exists\, n \in \mathbb{N}$ s.t. $a_i = b_i\ \forall i < n$, $a_n = 0$, $b_n = 1$ & $a_i = 1\ \forall i > n$ & $b_i = 0\ \forall i > n$.

e.g. $\bar{a} = 0\,0\,0\,1\,0\,1\,1\,1\,1 \ldots$

$\bar{b} = 0\,0\,0\,1\,1\,0\,0\,0\,0 \ldots$